Problem 1. Answer: the unit circle $|z| = 1$. □

Problem 2. Answer: $\int_{-\pi}^{\pi} \frac{1}{1 + \sin \theta} d\theta = \pi \sqrt{2}$. □

Problem 3. Answer: $\int_{-\infty}^{\infty} \frac{\cos(2x)}{(x^2 + 1)^2} dx = \frac{3\pi e^{-2}}{2}$. The function $f(z) = e^{2iz}/(z^2 + 1)^2$ has a double pole at $z = i$ with residue $3e^{-2}/(4i)$. □

Problem 4. The polynomial $f(z) = z^3 - z + 1$ has two zeros in the right half plane. Use Rouche’s theorem with $g(z) = z^3 + 1$: the inequality $|f(z) - g(z)| < |g(z)|$ holds on the imaginary axis and also for $|z|$ large. □

Problem 5. Answer: $f(z) = (z + 1)/(1 - z)$. The function $f$ can be computed using standard method of separation of variables for solving first order ordinary differential equations. □

Problem 6. Let $\alpha \in \mathbb{D}$, and consider the function

$$B_\alpha(z) = \frac{z - \alpha}{1 - \alpha z}.$$  

The function $B_\alpha$ has a single zero at the point $\alpha$ and is of constant modulus 1 on the unit circle.

The extremals of the problem have the form

$$f(z) = cB_{1/2}(z)B_{1/3}(z),$$

where $c$ is a constant of modulus 1. For such an $f$ we have that

$$|f(0)| = |B_{1/2}(0)||B_{1/3}(0)| = 1/6.$$ □