Problem 1. Passing to the complex coordinate we have that
\[ u(z) = \frac{(z - \bar{z})}{2iz\bar{z}} = \frac{1}{2i} \left( \frac{1}{z} - \frac{1}{\bar{z}} \right). \]
From this formula we see that \( u \) is the imaginary part of the function \( f(z) = -1/z \).
By a result from the course we conclude that \( u \) is harmonic in \( \Omega = \mathbb{C} \setminus \{0\} \). \( \square \)

Problem 2. By a well-known formula for a finite geometric sum we have that
\[ p(z) - p(a) = \frac{z^n - a^n}{z - a} = \sum_{k=0}^{n-1} z^k a^{n-1-k} \]
for \( z \neq a \). Passing to the limit letting \( z \to a \) we conclude that the complex derivative \( p'(a) \) exists and equals \( na^{n-1} \). \( \square \)

Problem 3. The searched for Laurent series expansion is
\[ k(z) = \frac{1}{(1 - z)^2} = -\sum_{m=-\infty}^{-2} (m + 1) z^m \]
for \( |z| > 1 \). \( \square \)

Problem 4. Set
\[ J(a) = \int_0^{2\pi} \frac{1}{a + \cos \theta} \, d\theta \]
for \( a > 1 \). Differentiating under the integral we have that
\[ J'(a) = -\int_0^{2\pi} \frac{1}{(a + \cos \theta)^2} \, d\theta = -I(a) \]
for \( a > 1 \). In Gamelin Section VII.3 it is shown that
\[ J(a) = \frac{2\pi}{\sqrt{a^2 - 1}} \]
for \( a > 1 \). By differentiation we have
\[ J'(a) = -\frac{2\pi a}{(a^2 - 1)\sqrt{a^2 - 1}}, \]
so that \( I(a) = 2\pi a/((a^2 - 1)\sqrt{a^2 - 1}) \) for \( a > 1 \). \( \square \)

Problem 5. Let \( f \) be an entire function satisfying (*) and consider its Taylor expansion
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}. \]
From the Cauchy estimates and the maximum principle we have that
\[ |a_n|r^n \leq \max_{|z|=r}|f(z)| \leq r + 1/r \]
for \( r > 0 \) and \( n = 0, 1, 2, \ldots \). Letting \( r \to +\infty \) we see that \( a_n = 0 \) for \( n \geq 2 \).
It remains to evaluate condition (*) on an affine function: \( f(z) = az + b \) with \( a, b \in \mathbb{C} \). It is straightforward to check that for such an \( f \) we have that
\[
\max_{|z|=r} |f(z)| = |a|r + |b|
\]
for \( r > 0 \). We are thus led to determine under what conditions on \( a, b \in \mathbb{C} \) the inequality
\[
(1) \quad |a|r + |b| \leq r + 1/r
\]
holds true for all \( r > 0 \). Letting \( r \to +\infty \) in (1) we see that \( |a| \leq 1 \). For \( |a| \leq 1 \) we have that (1) holds for \( r > 0 \) if and only if
\[
|b| \leq \inf_{r>0} \left( (1 - |a|)r + 1/r \right).
\]
A calculation shows that
\[
\inf_{r>0} \left( (1 - |a|)r + 1/r \right) = 2\sqrt{1 - |a|}
\]
for \( |a| \leq 1 \). We conclude that an entire function \( f \) satisfies (*) if and only if it has the form \( f(z) = az + b \) for some \( a, b \in \mathbb{C} \) such that \( |a| \leq 1 \) and \( |b| \leq 2\sqrt{1 - |a|} \). \( \square \)

**Problem 6.** Assume that the function \( f \) is not vanishing identically in \( \Omega \). Then there exists a disc \( B(a, r) \subset \Omega \) with positive radius \( r > 0 \) such that \( f(z) \neq 0 \) for \( z \in B(a, r) \). By division we see that \( g(z) = 0 \) for \( z \in B(a, r) \). Since \( g \) is analytic we conclude from connectedness of \( \Omega \) that \( g(z) = 0 \) for all \( z \in \Omega \) (see Gamelin Section V.7). \( \square \)

**Remark.** In algebraic terminology the result of Problem 6 says that the commutative ring of analytic functions \( H(\Omega) \) has no zero divisors when \( \Omega \) is connected.