Problem 1. It is evident that the relation \( \sim \) is an equivalence relation meaning that it is reflexive (\( z \sim z \) for every \( z \in U \)), symmetric (\( z \sim w \) implies \( w \sim z \) for all \( z, w \in U \)), and transitive (\( z \sim w \) and \( w \sim u \) implies \( z \sim u \) for all \( z, w, u \in U \)). The equivalence classes are
\[
\{1, -1\}, \quad \left\{ \frac{1}{2} + i \frac{\sqrt{3}}{2} , - \frac{1}{2} - i \frac{\sqrt{3}}{2} \right\} \quad \text{and} \quad \left\{ - \frac{1}{2} + i \frac{\sqrt{3}}{2} , \frac{1}{2} - i \frac{\sqrt{3}}{2} \right\}.
\]
Notice that the relation \( \sim \) identifies antipodal points on the unit circle. □

Problem 2. The solution of the recursion problem is:
\[
a_n = n^3 - 4n^2 + 3n + 1, \quad n = 0, 1, 2, \ldots.
\]
□

Problem 3. The solutions of the congruences are:
\[
x = 19 + 140n, \quad n \in \mathbb{Z}.
\]
□

Problem 4. We shall use the so-called principle of inclusion and exclusion. Let \( X \) be the set of all bijections on \( S \), and denote by \( X_k \) the subset of all \( \sigma \in X \) such that \( \sigma(k) = k \) (\( k = 1, 2, 3, 4 \)). Clearly
\[
|X| = 4! = 24 \quad \text{and} \quad |X_1 \cap X_2 \cap X_3 \cap X_4| = 1.
\]
A calculation gives that \( |X_1| = |X_2| = |X_3| = |X_4| = 3! = 6 \), \( |X_j \cap X_k| = 2! = 2 \) for \( j \neq k \), \( |X_j \cap X_k \cap X_l| = 1 \) for \( j \neq k \), \( k \neq l \) and \( j \neq l \).

By the above mentioned principle we have that
\[
|X \setminus \bigcup_{j=1}^4 X_j| = |X| - 4|X_1| + \binom{4}{2} |X_1 \cap X_2| - \binom{4}{3} |X_1 \cap X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3 \cap X_4| = 9.
\]
□

Problem 5. The polynomial \( p \) factorizes as
\[
p(x) = (x^2 + x + 1)^2
\]
in \( \mathbb{Z}_2[x] \). Therefore the quotient ring \( R = \mathbb{Z}_2[x]/(p(x)) \) is not a field. A calculation gives that
\[
x^3 + 1 = (x + 1)(x^2 + x + 1)
\]
in \( \mathbb{Z}_2[x] \). Therefore the element \([x^3 + 1]\) is not invertible in \( R \). □

Problem 6. Recall the recursion formula for Stirling numbers of the second kind
\[
S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)
\]
for \( n, k \geq 2 \), and that \( S(n, 1) = 1 = S(n, n) \) for all \( n \geq 1 \). By this we have that

\[
S_n(x) = x + x^n + \sum_{k=2}^{n-1} S(n-1, k-1)x^k + \sum_{k=2}^{n-1} kS(n-1, k)x^k
\]

\[
= \sum_{k=2}^{n} S(n-1, k-1)x^k + \sum_{k=1}^{n-1} kS(n-1, k)x^k = xS_{n-1}(x) + xS'_{n-1}(x).
\]

This gives the recursion formula for the polynomials \( S_n \) stated in the problem. The fact that \( S_1(x) = x \) follows from \( S(1, 1) = 1 \). Furthermore,

\[
S_2(x) = x + x^2, \quad S_3(x) = x + 3x^2 + x^3 \quad \text{and} \quad S_4(x) = x + 7x^2 + 6x^3 + x^4.
\]