Problem 1. The Möbius transformation
\[ f(z) = \frac{z + 1 + i}{z - 1} \]
solves the mapping problem. Recall a result about Möbius transformations saying that given two triples \((z_1, z_2, z_3)\) and \((w_1, w_2, w_3)\) of points in the extended complex plane \(\mathbb{C}_\infty\) there exists exactly one Möbius transformation \(f\) such that \(f(z_j) = w_j\) for \(j = 1, 2, 3\).

Problem 2. A differentiation gives that
\[ \frac{\partial p}{\partial \bar{z}} = 6|z|^4 - 18|z|^2 + 12 = 6(|z|^2 - 1)(|z|^2 - 2). \]
By a standard result the complex derivative \(p'(z_0)\) exists if and only if \(|z_0| = 1\) or \(|z_0| = \sqrt{2}\).

Problem 3. The function \(f\) is given by
\[ f(z) = \frac{1}{(z - 1)^2} + \frac{1}{z + 1}, \quad z \in \mathbb{C} \setminus \{-1, 1\}. \]
The Laurent expansion around the point 1 is
\[ f(z) = \frac{1}{(z - 1)^2} + \sum_{k=0}^{\infty} (-1)^k 2^{k-1} (z - 1)^k, \quad z \in \Omega. \]

Problem 4. First since the integrand is an even function we have that
\[ I = \int_0^\pi \frac{\cos^2 \theta + 1}{\cos \theta + 2} \, d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos^2 \theta + 1}{\cos \theta + 2} \, d\theta. \]
Next observe that a polynomial division gives
\[ \frac{\cos^2 \theta + 1}{\cos \theta + 2} = \cos \theta - 2 + \frac{5}{\cos \theta + 2}, \]
so that
\[ \int_{-\pi}^{\pi} \frac{\cos^2 \theta + 1}{\cos \theta + 2} \, d\theta = -4\pi + 5 \int_{-\pi}^{\pi} \frac{d\theta}{\cos \theta + 2}. \]
In the course is was shown that
\[ \int_{-\pi}^{\pi} \frac{d\theta}{\cos \theta + a} = \frac{2\pi}{\sqrt{a^2 - 1}} \]
for \(a > 1\) (see Gamelin textbook Section VII.3). We now conclude that
\[ I = \frac{1}{2} \left( -4\pi + 5 \cdot \frac{2\pi}{\sqrt{3}} \right) = \left( \frac{5\sqrt{3}}{3} - 2 \right)\pi. \]
Problem 5. Consider the function
\[ f(z) = \frac{1 - e^{iz}}{z^2}. \]
The function \( f \) is analytic except for a simple pole at the origin. Denote by \( \gamma_r \) a half-circle \( |z| = r \) from \( r \) to \(-r \) in the upper half-plane, and write \([a, b]\) for a line segment from \( a \) to \( b \). Let \( 0 < \varepsilon < 1 < R < +\infty \). By Cauchy’s theorem we have that
\[
\int_{-\gamma_R} f(z)dz = 0.
\]
Notice that \( \int_{\gamma_R} f(z)dz \to 0 \) as \( R \to +\infty \). Also
\[
\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)dz = \pi i \text{Res}(f; 0)
\]
by the fractional residue theorem (see Gamelin textbook Section VII.5). Observe also that
\[
\int_{\varepsilon \leq |z| \leq R} f(x)dx = 2 \int_{\varepsilon}^{R} \frac{1 - \cos x}{x^2} dx
\]
by standard facts about odd and even functions. Letting \( \varepsilon \to 0^+ \) and \( R \to +\infty \) in (1) we have that
\[
2 \int_{0}^{\infty} \frac{1 - \cos x}{x^2} = \pi i \text{Res}(f; 0).
\]
The residue at the origin is easily calculated as \( \text{Res}(f; 0) = -i \). We conclude that
\[
\int_{0}^{\infty} \frac{1 - \cos x}{x^2} = \frac{\pi}{2}.
\]
□

Problem 6. Assume first that the function \( u \) is continuous up to the boundary \( T = \partial\mathbb{D} \). By the Poisson integral formula we have that
\[
u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(ze^{-i\theta})u(ze^{i\theta})d\theta, \quad z \in \mathbb{D},
\]
where
\[
P(z) = \frac{1 - |z|^2}{|1 - z|^2}, \quad z \in \mathbb{D},
\]
is the Poisson kernel. An application of the triangle inequality gives that
\[
\frac{1 - |z|}{1 + |z|} \leq P(z) \leq \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D}.
\]
An estimation using (2) and (3) gives that
\[
\frac{1 - |z|}{1 + |z|} \leq u(z) \leq \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D},
\]
if \( u \) is also continuous in the closed disc \( \overline{\mathbb{D}} \).

Let us now deduce the general case. Let \( 0 < r < 1 \) and consider the dilated function \( u_r \) defined by
\[
u_r(z) = u(rz), \quad z \in \mathbb{D}.
\]
By (4) applied to the function $u_r$ we have that
\[
\frac{1 - |z|}{1 + |z|} \leq u_r(z) \leq \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D}.
\]

Now letting $r \to 1$ inequality (4) follows for a general non-negative harmonic function $u$ in $\mathbb{D}$. \hfill \Box

Remark. The inequality in Problem 6 is known in the literature as Harnack’s inequality.