Problem 1. The Möbius transformation

\[ f(z) = \frac{(3 + i)z - 2}{z - 1 + i} \]

solves the mapping problem. Recall that Möbius transformations map the class of circles/lines onto itself and preserve orientation. Using these facts we have that \( f(D) \) is the half-plane

\[ f(D) = \{ w = u + iv \in \mathbb{C} : v > 2u \}. \]

Problem 2. A calculation gives that

\[ i \frac{z}{z} = \frac{y}{x^2 + y^2} + i \frac{x}{x^2 + y^2}, \]

where \( z = x + iy \neq 0 \). The function

\[ u(x, y) = \frac{y}{x^2 + y^2} \]

is harmonic being the real part of an analytic function. The normalized harmonic conjugate is given by

\[ v(x, y) = \frac{x}{x^2 + y^2}. \]

Problem 3. Since the integrand is an even function we have that

\[ \int_0^{\infty} \frac{x \sin(ax)}{(1 + x^2)^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin(ax)}{(1 + x^2)^2} dx. \]

Next an integration by parts gives that

\[ \int_{-\infty}^{\infty} \frac{x \sin(ax)}{1 + x^2} dx = \frac{a}{2} \int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2} dx. \]

For the calculation of the latter integral we refer to Gamelin pp. 201 to the extent that

\[ \int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2} dx = \pi e^{-a}. \]

Going back to our original integral we have that

\[ \int_0^{\infty} \frac{x \sin(ax)}{(1 + x^2)^2} dx = \frac{\pi e^{-a}}{4}. \]

Problem 4. We shall apply Rouche’s theorem. Let \( g(z) = 3z^3 \). For \( |z| = 1 \) we have

\[ |g(z) - p(z)| = |z + 1 + i| \leq |z| + |1 + i| = 1 + \sqrt{2} < 3 = |g(z)|, \]

showing that the assumption of Rouche’s theorem is fulfilled. We now conclude that the functions \( g \) and \( p \) have the same number of zeros in \( D \), namely, 3.
**Problem 5.** The differential equation is of separable type. Rewrite the equation as
\[ \exp(-f(z))f'(z) = z. \]
Notice that \((-\exp(-f))' = \exp(-f)f'\), which by a well-known theorem gives that
\[ -\exp(-f(z)) = z^2/2 + c \]
for some constant \(c\). The initial data \(f(0) = 0\) gives that \(c = -1\). Thus,
\[ -\exp(-f(z)) = z^2/2 - 1 \]
for \(z\) near the origin. Solving for \(f(z)\) we have that
\[ f(z) = -\log(1 - z^2/2), \]
which is the solution of our initial value problem. Here \(\log\) is the usual principal branch logarithm.

To find the power series expansion around the origin we differentiate and expand in a geometric series
\[
f'(z) = \frac{z}{1 - z^2/2} = z \sum_{k=0}^{\infty} 2^{-k} z^{2k} = \sum_{k=0}^{\infty} 2^{-k} z^{2k+1},
\]
which gives that
\[
f(z) = \sum_{k=0}^{\infty} \frac{z^{2k+2}}{(2k+2)2^k}.
\]
The radius of convergence of this power series expansion is \(\sqrt{2}\). \(\square\)

**Problem 6.** For \(\alpha \in \mathbb{D}\) we consider an automorphism of the unit disc of the form
\[
\varphi_{\alpha}(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}.
\]
This function \(\varphi_{\alpha}\) has the property that it is self-involutive \(\varphi_{\alpha} \circ \varphi_{\alpha} = \text{id}\), and its derivative is given by
\[
\varphi'_{\alpha}(z) = -\frac{1 - |\alpha|^2}{(1 - \overline{\alpha}z)^2}.
\]

Let now \(f\) be analytic in \(\mathbb{D}\) with \(\sup_{z \in \mathbb{D}} |f(z)| \leq 1\) and \(f(0) = 1/2\), and consider the function
\[
g = \varphi_{1/2} \circ f(z) = \varphi_{1/2}(f(z)), \quad z \in \mathbb{D}.
\]
This function \(g\) is then such that \(\sup_{z \in \mathbb{D}} |g(z)| \leq 1\) and \(g(0) = 0\). Calculating \(g'(0)\) we see that
\[
g'(0) = \varphi'_{1/2}(f(0))f'(0) = \varphi'_{1/2}(1/2)f'(0) = -\frac{4}{3}f'(0).
\]
From this last calculation we have that \(f'(0) = 3/4\) if and only if \(g'(0) = -1\).

By previous considerations we are led to set \(g(z) = -z\) which gives that
\[
f(z) = \varphi_{1/2} \circ g(z) = \frac{z + 1/2}{1 + z/2}.
\]
By considerations above, this function \(g\) has all the properties quoted in Problem 6. Notice also that by Schwarz lemma this function \(g\) is the only analytic function in \(\mathbb{D}\) with these properties. \(\square\)
Remark. The interpolation problem in Problem 6 is usually said to be of Caratheodory type. Similar interpolation problems have been much studied, and have attracted interest also from the point of view of signal processing.