Problem 1. The function
\[ g(z) = 2z + 1 + 2i \]
maps \( \mathbb{D} \) bijectively onto \( D \) but does not satisfy the desired normalization conditions. We shall modify \( g \) correspondingly.

Recall that for \( \alpha \in \mathbb{D} \) the function
\[ \varphi_{\alpha}(z) = \frac{\alpha - z}{1 - \overline{\alpha}z} \]
maps \( \mathbb{D} \) conformally onto itself. Also \( \varphi_{\alpha}(0) = \alpha \). The derivative of \( \varphi_{\alpha} \) is given by
\[ \varphi'_{\alpha}(z) = -\frac{1 - |\alpha|^2}{(1 - \overline{\alpha}z)^2}. \]

Notice that \( g(-i/2) = 1 + i \). The function
\[ h = g \circ \varphi_{-i/2} \]
maps \( \mathbb{D} \) conformally onto \( D \) and also satisfies that \( h(0) = 1 + i \). The same is true for each of the functions
\[ f e^{i\theta}(z) = g \circ \varphi_{-i/2}(e^{i\theta} z), \]
where \( e^{i\theta} \in \mathbb{T} = \partial \mathbb{D} \) (we compose with a rotation of the disc).

We shall next determine \( e^{i\theta} \in \mathbb{T} \) such that \( f'_{e^{i\theta}}(0) > 0 \). Differentiating we have that
\[ f'_{e^{i\theta}}(0) = g'(-i/2)\varphi'_{-i/2}(0)e^{i\theta} = -\frac{3}{2}e^{i\theta}. \]

We are now led to set
\[ f(z) = f_{-1}(z) = \frac{2 + 2i + (2 + i)z}{2 + iz}. \]
This function \( f \) maps \( \mathbb{D} \) conformally onto \( D \) and satisfies that \( f(0) = 1 + i \) and \( f'(0) = 3/2 > 0. \)

Problem 2. By standard theory the function \( f \) has complex derivative at a point \( z_0 \) if and only if the Cauchy-Riemann equation \( \partial f / \partial \overline{z} = 0 \) holds at that point \( z_0 \). By differentiation
\[ \frac{\partial f}{\partial \overline{z}} = -z^2 \sin(z^2 \overline{z}). \]
A calculation gives that \( f \) has complex derivative at a point \( z_0 \) if and only if \( z_0 \) is of the form \( z_0 = \pm (\pi n)^{1/3} \) for some non-negative integer \( n \).

Problem 3. Setting \( z = e^{i\theta}, -\pi < \theta < \pi, \) we rewrite the given integral as a line integral along the unit circle as
\[ \int_{-\pi}^{\pi} \frac{1}{3 + \sin \theta + \cos \theta} d\theta = \frac{2}{1 + i} \int_{|z|=1} f(z)dz, \]
where
\[ f(z) = \frac{1}{z^2 + 3(1 + i)z + i}. \]
The rational function $f$ has simple poles at the points where
$$z^2 + 3(1 + i)z + i = 0.$$ 
A calculation gives that the poles of $f$ are
$$z_1 = (-\frac{3}{2} + \frac{\sqrt{7}}{2})(1 + i) \quad \text{and} \quad z_2 = -(\frac{3}{2} + \frac{\sqrt{7}}{2})(1 + i).$$
Using that $\sqrt{7} < 3$, we have
$$|z_2| = \frac{3 + \sqrt{7}}{2} \sqrt{2} > \sqrt{7}\sqrt{2} = \sqrt{14} > 1,$$
showing that $z_2$ is located outside of the unit disc. Similarly, the calculation
$$|z_1| = \frac{3 - \sqrt{7}}{2} \sqrt{2} = \frac{\sqrt{2}}{3 + \sqrt{7}} < \frac{1}{\sqrt{14}} < 1,$$
shows that $z_1$ is located in $D$. Applying the residue theorem, we have that
$$\int_{|z|=1} f(z)dz = 2\pi i \operatorname{Res}(f; z_1).$$
The residue of $f$ at $z_1$ is computed as
$$\operatorname{Res}(f; z_1) = \left[\frac{z - z_1}{f(z)}\right]_{z=z_1} = \frac{1}{z_1 - z_2} = \frac{1}{\sqrt{7}(1 + i)}.$$
Going back to our original integral we have that
$$\int_{-\pi}^{\pi} \frac{1}{3 + \sin \theta + \cos \theta} d\theta = \frac{2\pi i}{\sqrt{7}(1 + i)} = \frac{2\pi}{\sqrt{7}}.$$

**Remark Problem 3.** Applying a suitable rotation of the circle (phase angle) we can reduce the calculation of the given integral to that of computing an integral of the form
$$\int_{-\pi}^{\pi} \frac{1}{a + \sin \theta} d\theta \quad \text{or} \quad \int_{-\pi}^{\pi} \frac{1}{a + \cos \theta} d\theta,$$
where $a$ is a constant. Indeed,
$$\int_{-\pi}^{\pi} \frac{1}{3 + \sin \theta + \cos \theta} d\theta = \int_{-\pi}^{\pi} \frac{1}{3 + \sqrt{2}\cos \theta} d\theta = \frac{2\pi}{\sqrt{7}},$$
where the last equality follows by the calculation in Gamelin Section VII.3.

**Problem 4.** The integral is easily computed without reference to residue theory. By a change of variables we have
$$\int_{0}^{\infty} \frac{\log x}{2 + x^2} dx = \frac{\sqrt{2}}{2} \int_{0}^{\infty} \frac{\log(\sqrt{2}x)}{1 + x^2} dx = \frac{\sqrt{2}}{2} \int_{0}^{\infty} \frac{\log x}{1 + x^2} dx + \frac{\pi \sqrt{2} \log 2}{8},$$
using that $d\arctan x/\sqrt{2} = 1/(1 + x^2)$. Also, by a change of variables we see that
$$\int_{0}^{\infty} \frac{\log x}{2 + x^2} dx = \int_{0}^{1} \frac{\log x}{1 + x^2} dx + \int_{1}^{\infty} \frac{\log x}{1 + x^2} dx = \int_{0}^{1} \frac{\log(x)}{1 + x^2} dx - \int_{0}^{1} \frac{\log(x)}{1 + x^2} dx = 0.$$
We conclude that
$$\int_{0}^{\infty} \frac{\log x}{2 + x^2} dx = \frac{\pi \sqrt{2} \log 2}{8}.$$
Problem 5. Consider the geometric series

\[ \frac{1}{1 - z} = \sum_{k=0}^{\infty} z^k, \quad |z| < 1. \]

Differentiating and multiplying by \( z \) we find that

\[ \frac{z}{(1 - z)^2} = \sum_{k=1}^{\infty} k z^k, \quad |z| < 1. \]

Differentiating and multiplying by \( z \) once again we find that

\[ \frac{z(z + 1)}{(1 - z)^3} = \sum_{k=1}^{\infty} k^2 z^k \]

for \( |z| < 1 \). It is now apparent that the formula

\[ f(z) = \frac{z(z + 1)}{(1 - z)^3} + 5 \frac{z}{(1 - z)^2} - \frac{1}{1 - z} \]

continues \( f \) analytically to \( \mathbb{C} \setminus \{1\} \). The Laurent expansion of \( f \) around the point 1 is given by

\[
\begin{align*}
    f(z) &= -\frac{z^2 + z}{(z - 1)^3} + 5 \frac{z}{(z - 1)^2} + \frac{1}{z - 1} \\
    &= \frac{((-z - 1)^2 + (z - 1) + 1)}{(z - 1)^3} + 5 \frac{(z - 1) + 1}{(z - 1)^2} + \frac{1}{z - 1} \\
    &= -\frac{(z - 1)^2 + 2(z - 1) + 1 + (z - 1) + 1}{(z - 1)^3} + 5 \left( \frac{1}{z - 1} + \frac{1}{(z - 1)^2} \right) + \frac{1}{z - 1} \\
    &= -\left( \frac{1}{z - 1} + 3 \frac{1}{(z - 1)^2} + 2 \frac{1}{(z - 1)^3} \right) + 5 \frac{1}{(z - 1)^2} + 6 \frac{1}{z - 1} \\
    &= 5 \frac{1}{z - 1} + 2 \frac{1}{(z - 1)^2} - 2 \frac{1}{(z - 1)^3}.
\end{align*}
\]

□

Problem 6. A calculation gives that

\[ P(z) = \frac{1 - |z|^2}{|1 - z|^2} - \frac{1}{1 - z} + \frac{1}{1 - z} - 1 = 2 \text{Re} \left( \frac{1}{1 - z} \right) - 1. \]

Being the real part of an analytic function, the function \( P \) is harmonic. Also, the normalized harmonic conjugate is the function

\[ Q(z) = 2 \text{Im} \left( \frac{1}{1 - z} \right) = \frac{2 \text{Im} z}{|1 - z|^2}. \]

□

Remark. The function \( P \) is also known as the Poisson kernel for the unit disc \( \mathbb{D} \).