ASYMPTOTICS FOR A BETA FUNCTION ARISING IN POTENTIAL THEORY

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Abstract. We consider a family of incomplete Beta functions arising in the study of Green functions for related weighted Laplacians. An approximation scheme of Taylor type gives rise to associated relative remainder terms. We study the (scaled) limit behavior of these relative remainder terms.

0. Introduction

Let \( \alpha > -1 \) be a real number. The function \( h_\alpha \) is given by

\[
h_\alpha(x) = -\int_x^1 \frac{(1-t)^\alpha}{t} \, dt
\]

for \( 0 < x < 1 \). The assumption \( \alpha > -1 \) is needed for the integral in (0.1) to exist. The function \( h_\alpha \) has close relationship to the family of Beta functions. In fact,

\[
h_\alpha(x) = -B(1-x; \alpha + 1, 0)
\]

for \( 0 < x < 1 \), where

\[
B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1} \, dt
\]

is the incomplete Beta function (see Proposition 1.1). It is natural to compare the function \( h_\alpha \) with the usual natural logarithm which is obtained for \( \alpha = 0 \) (see Proposition 1.2). However, the function \( h_\alpha \) behaves like a constant multiple of \( (1-x)^{\alpha+1} \) as \( x \to 1 \). This paper is concerned with a certain approximation scheme for the function \( h_\alpha \) which takes this latter boundary behavior into account.

Our interest in the function \( h_\alpha \) first arose in the study of potential theory of weighted Laplacians of the form

\[
\Delta_{\alpha; z} = \Delta_{D; \alpha; z} = \partial_z (1-|z|^2)^{-\alpha} \bar{\partial}_z, \quad z \in D,
\]

in the open unit disc \( D \) in the complex plane \( \mathbb{C} \). Here \( \partial \) and \( \bar{\partial} \) are the usual complex partial derivatives and \( \alpha > -1 \) is a weight parameter. In this context the function \( z \mapsto h_\alpha(|z|^2) \) plays a role of a natural fundamental solution for \( \Delta_{\alpha} \) located at the origin. As our study progressed there appeared a need to approximate and efficiently estimate the function \( h_\alpha \). A starting point for the present paper was to further study the approximation scheme for the function \( h_\alpha \) used in Olofsson [20, Lemma 5.1] for such a purpose.

Date: June 24, 2019.
2010 Mathematics Subject Classification. Primary: 33B20; Secondary: 30B99, 31A10.
Key words and phrases. Relative remainder term, incomplete Beta function, Green’s function, weighted Laplace operator.
Let us return to the function $h_\alpha$ for a general parameter $\alpha > -1$. We set

$$f_1(x) = \log(x) \text{ for } x > 0$$

for $k \geq 2$, where prime (') denotes derivative. Repeated integration by parts in formula (0.1) (differentiating on the factor $(1-t)^\alpha$) leads to an expansion of the form

$$h_\alpha(x) = \sum_{n=1}^{k} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 2 - n)} f_n(x) + R_{\alpha,k}(x)$$

for $0 < x < 1$ and $k \geq 1$, where $\Gamma$ is the standard Gamma function and the remainder term $R_{\alpha,k}$ is determined by (0.3) (see Theorem 2.1). Primitives involving logarithms is an established topic in the calculus community. For the sake of completeness we supply an explicit formula for the function $f_k$ (see Theorem 2.2). We provide several formulas for the remainder term $R_{\alpha,k}$, notably,

$$R_{\alpha,k}(x) = (-1)^k \frac{\sin(\alpha \pi)}{\pi (\alpha + 1 - k)(k-1)} \int_0^1 (1-t)^{k-1} g_\alpha((1-x)t) \, dt$$

for $0 < x < 1$, where

$$g_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\alpha + 1 + n}, \quad 0 < x < 1$$

(see Theorem 3.2). We think of the expansion (0.3) as a parallel of Taylor's formula adapted to the present context.

Of particular interest is the limit behavior of the remainder term $R_{\alpha,k}$ as $k$ gets big. We show that

$$\lim_{k \to +\infty} k^{\alpha+1} R_{\alpha,k}(x) = \frac{-1}{(\alpha + 1)} \sin(\alpha \pi)$$

uniformly for $0 < x < 1$ (see Corollary 4.1). This result is a consequence of a more general result on the limit behavior of a certain relative remainder term $E_{\alpha,k}$ derived from the function $R_{\alpha,k}$.

It turns out that the function $g_\alpha$ appearing in (0.4) continues to an analytic function in the slit plane $\mathbb{C} \setminus [1, \infty)$ (see Lemma 3.1). We are thus led to consider the function

$$E_{\alpha,k}(z) = k \int_0^1 (1-t)^{k-1} g_\alpha((1-z)t) \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0]$$

for $k = 1, 2, \ldots$. Observe that $E_{\alpha,k}$ is analytic in the slit plane $\mathbb{C} \setminus (-\infty, 0]$ as this property is inherited from the function $g_\alpha$ being analytic in $\mathbb{C} \setminus [1, \infty)$. We call $E_{\alpha,k}$ the relative remainder term because of its relationship to the remainder term $R_{\alpha,k}$ dictated by (0.4). The main results of this paper concern the limit behavior of this relative remainder term. We show that

$$\lim_{k \to +\infty} k^j E^{(j)}_{\alpha,k}(x) = (-1)^j (j!)^2 \frac{(\alpha + 1 + j)}{\alpha + 1 + j}$$

for every $j \geq 0$ with uniform convergence on every bounded interval of the form $0 < x < A$, where $A > 0$ is a finite constant (see Theorem 4.1). Here, as usual,
$E_{\alpha,k}^{(j)}$ is the $j$-th derivative of the function $E_{\alpha,k}$. As a byproduct of (0.6) we obtain (0.5).

We then turn our attention to the scaled limit of the relative remainder term. We show that

$$(0.7) \quad \lim_{k \to +\infty} E_{\alpha,k}(kz) = \int_0^{\infty} e^{-t} g_{\alpha}(-zt) \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

with uniform convergence on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ (see Theorem 5.2). Since the limit in (0.7) holds in the sense of normal convergence of analytic functions, it contains also the corresponding limit assertions for higher order derivatives. As a consequence, since the limit in (0.7) is not a polynomial, we see that the convergence in (0.6) is not uniform over the whole half-axis $0 < x < +\infty$.

Behind our results (0.6) and (0.7) discussed above lies an analysis of the limit behavior of the Beta distribution. The limit (0.6) is proved by a summability kernel argument (see Lemmas 4.1 and 4.2). Concerning the scaled limit, we first show that (0.7) holds on the positive half-axis, that is, at points where $z = x > 0$ (see Theorem 5.1). A normal family argument then establishes the general case (see Theorem 5.2).

We wish to point out also that the limit

$$(0.8) \quad L_{\alpha}(z) = \int_0^{\infty} e^{-t} g_{\alpha}(-zt) \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

in (0.7) has a close relation to the so-called exponential integral defined by

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where the integration is from $z$ to infinity along a ray parallel to the real axis and of increasing real part. We show that

$$(0.8) \quad L_{\alpha}(z) = \frac{1}{z^{\alpha+1}} \int_{1/z}^{\infty} t^{-(\alpha+1)} e^t E_1(t) \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where the integration is as above (see Theorem 5.3). The power in (0.8) is defined in the usual way using a logarithm in $\mathbb{C} \setminus (-\infty, 0]$ which is real on the positive real axis. Of mention here is an estimate of the function $E_1$ in the open right half-plane which is due to Gautschi [11] (see Lemma 5.5). This latter estimate ensures that the integral in (0.8) is absolutely convergent.

A classical theorem of Hurwitz describes the location of zeroes for a normally convergent sequence of analytic functions (see for instance [9, Section VIII.3]). Our result (0.7) on the scaled limit thus relates to investigations of Gabor Szegő [22], Dieudonné [6] and later Newman and Rivlin [16, 17] on partial sums of the exponential function; see also Edrei, Saff and Varga [7]. We mention here also recent work by Ameur, Kang and Makarov [2, Section 7.5] and Hedennalm and Wennman [14].

The function $h_{\alpha}$ plays an important role in the study of potential theory of weighted Laplacians of the form

$$(0.9) \quad \Delta_{\Omega,\alpha} z = \partial_z w_{\Omega,\alpha}(z)^{-1} \partial_{z'}, \quad z \in \Omega,$$

where $\partial_z$ and $\partial_{z'}$ are the usual complex partial derivatives and $\Omega$ is a simply connected planar domain not equal to the whole complex plane. The function
\( w_{\Omega, \alpha} : \Omega \to (0, \infty) \) appearing in (0.9) is a so-called standard weight function in \( \Omega \) with weight parameter \( \alpha \) which is defined by
\[
  w_{\Omega, \alpha}(z) = K_{\Omega}(z, z)^{-\alpha/2}, \quad z \in \Omega,
\]
where \( K_{\Omega} \) is the Bergman kernel for \( \Omega \) relative to usual Lebesgue area measure \( dA(z) = dx dy / \pi, \quad z = x + iy \), normalized so that the open unit disc \( D \) has unit area. For the parameter value \( \alpha = 0 \) one obtains in this way the usual Laplacian \( \Delta_{\Omega;0} = \partial \overline{\partial} \) normalized by a factor \( 1/4 \).

The weighted Laplacians \( \Delta_{\Omega, \alpha} \) turn out to have an important property of conformal invariance with respect to weighted compositions with conformal maps. A further analysis along these lines leads to the formula
\[
  G_{\Omega, \alpha}(z, \zeta) = K_{\Omega}(z, \zeta)^{-\alpha/2} h_{\alpha}(\rho_{\Omega}(z, \zeta)^2)
\]
for the Green function for \( \Delta_{\Omega, \alpha} \) valid when \( \alpha > -1 \), where \( K_{\Omega} \) is the Bergman kernel for \( \Omega \) and \( \rho_{\Omega} \) is the pseudo hyperbolic metric for \( \Omega \), respectively. The role played by the function \( h_{\alpha} \) in formula (0.10) is that of a fundamental solution at the origin: The function
\[
  u(z) = G_{D, \alpha}(z, 0) = h_{\alpha}(|z|^2), \quad z \in D \setminus \{0\},
\]
has the property that \( \Delta_{\alpha} u = \delta_0 \) in \( D \) in a distributional sense, where \( \Delta_{\alpha} \) is as in (0.2) and \( \delta_0 \) is a unit Dirac mass placed at the origin. The exact rate of decay of the function \( h_{\alpha}(x) \) as \( x \to 1 \) is important since it determines when the associated Green potentials exist. For a full proof of formula (0.10) we refer to Olofsson [20].

A notable predecessor is Gustav Behm [4] who established (0.10) in the important special case of the open unit disc \( D \). Earlier related papers are Garabedian [10], Olofsson and Wittsten [18], Olofsson [19] and Borichev and Hedenmalm [5].

1. First Observations

The function
\[
  B(x; a, b) = \int_0^x t^{a-1} (1-t)^{b-1} \, dt
\]
is known in the literature as the incomplete Beta function (see [1, Section 6.6] or [12, Chapter 1]).

**Proposition 1.1.** Let \( \alpha > -1 \) and let \( h_{\alpha} \) be as in (0.1). Then
\[
  h_{\alpha}(x) = -B(1-x; \alpha + 1, 0)
\]
for \( 0 < x < 1 \).

**Proof.** By the change of variables \( y = 1-t \) we have that
\[
  h_{\alpha}(x) = -\int_x^1 \frac{(1-t)^{\alpha}}{t} \, dt = -\int_0^{1-x} \frac{y^{\alpha}}{1-y} \, dy = -B(1-x; \alpha + 1, 0)
\]
for \( 0 < x < 1 \). \( \square \)

A comparison with the logarithm suggests the formula
\[
  h_{\alpha}(x) = \log(x) + \int_x^1 \frac{1-(1-t)^{\alpha}}{t} \, dt
\]
for \( 0 < x < 1 \). Formula (1.1) is straightforward to check.
Proposition 1.2. Let $\alpha > -1$ and let $h_\alpha$ be as in (0.1). Then

$$h_\alpha(x) = \log(x) + \sum_{k=1}^{\infty} \frac{1}{k} (1-x)^k - (1-x)^\alpha \sum_{k=1}^{\infty} \frac{1}{k+\alpha} (1-x)^k$$

for $0 < x < 1$.

Proof. We consider the integral

$$I_\alpha(x) = \int_x^1 \frac{1 - (1-t)^\alpha}{t} dt, \quad 0 < x < 1,$$

appearing in (1.1). Expanding the denominator in a geometric series we have that

$$I_\alpha(x) = \int_x^1 \frac{1}{1 - (1-t)^\alpha} dt = \sum_{k=0}^{\infty} \frac{1}{k+1} (1-x)^{k+1} - \sum_{k=0}^{\infty} \frac{1}{\alpha + k + 1} (1-x)^{\alpha+k+1}$$

for $0 < x < 1$ from which the result follows. \qed

Observe that the result of Proposition 1.2 simplifies to

$$h_\alpha(x) = \log(x) + \sum_{k=1}^{\alpha} \frac{1}{k} (1-x)^k$$

for $x > 0$ when $\alpha$ is a non-negative integer.

The Gamma function is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

for $\text{Re}(z) > 0$. Clearly $\Gamma(1) = 1$ and an integration by parts gives the familiar formula that $\Gamma(z+1) = z\Gamma(z)$. The function $\Gamma$ then continues to a meromorphic function in the complex plane with simple poles at the non-positive integers $0, -1, -2, \ldots$.

In the following pages the Gamma function will appear frequently, especially in the form of a quotient,

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)}$$

for some integer $k$ and $\alpha > -1$. This expression can be rewritten using Euler’s reflection formula,

$$(1.2) \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

(see [3, Theorem 1.2.1]), so that

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} = (-1)^{k+1} \frac{\sin(\alpha\pi)}{\pi} \Gamma(\alpha+1)\Gamma(k-\alpha)$$

for integer $k$. 
2. An expansion of the function $h_\alpha$

The purpose of this section is to set up some basics of an approximation scheme for the function $h_\alpha$ defined by (0.1). We proceed to study some related functions $\{f_k\}_{k=1}^\infty$.

We set $f_1(x) = \log(x)$ for $x > 0$ and

\begin{equation}
(2.1) \quad f_k(x) = \int_1^x f_{k-1}(t) \, dt, \quad x > 0,
\end{equation}

for $k \geq 2$. Observe that $f_{k+1}(1) = f_k(1) = 1$. Let $\alpha > -1$ and $k \geq 1$ we consider the function

\begin{equation}
(2.2) \quad R_{\alpha,k}(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} \int_x^1 (1-t)^{\alpha-k} f_k(t) \, dt
\end{equation}

for $0 < x < 1$. Existence of the integral in (2.2) for $\alpha > -1$ follows from Lemma 2.1.

The next result gives a basic approximation scheme for the function $h_\alpha$.

\begin{thm}
Let $\alpha > -1$ and $k \geq 1$. Let $h_\alpha$ be as in (0.1). Then

\begin{equation}
(2.3) \quad h_\alpha(x) = \sum_{n=1}^k \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2-n)} (1-x)^{\alpha+1-n} f_n(x) + R_{\alpha,k}(x)
\end{equation}

for $0 < x < 1$, where the functions $f_n$ and $R_{\alpha,k}$ are as in (2.1) and (2.2), respectively.
\end{thm}

\begin{proof}
We start with a preliminary observation. Let $0 < c < 1$. Partial integration gives that

\begin{equation}
\int_x^c (1-t)^{\alpha+1-k} f'_k(t) \, dt = (1-c)^{\alpha+1-k} f_k(c) - \int_c^x (1-t)^{\alpha-k} f_k(t) \, dt
\end{equation}

for $0 < x < 1$ and $k \geq 1$. From Lemma 2.1 we know that $f_k(t)(1-t)^{-k}$ is bounded as $t \to 1$. Hence we can let $c$ tend to 1 and obtain

\begin{equation}
(2.4) \quad - \int_x^1 (1-t)^{\alpha+1-k} f'_k(t) \, dt = (1-x)^{\alpha+1-k} f_k(x) - (\alpha+1-k) \int_x^1 (1-t)^{\alpha-k} f_k(t) \, dt
\end{equation}

for $0 < x < 1$ and $k \geq 1$.

We now prove formula (2.3) by induction on $k \geq 1$. We have that

\begin{equation}
\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} \int_x^1 (1-t)^{\alpha-1} f_k(t) \, dt
\end{equation}

for $0 < x < 1$ and $k \geq 1$. We have that

\begin{equation}
\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} \int_x^1 (1-t)^{\alpha-1} f_k(t) \, dt
\end{equation}

for $0 < x < 1$ and $k \geq 1$. We have that
where the last equality follows by (2.4). This proves (2.3) for \( k = 1 \).

Assume next that formula (2.3) holds for some \( k = p \geq 1 \). We proceed to prove that formula (2.3) holds for \( k = p + 1 \). Observe that

\[
R_{\alpha,p}(x) = -\frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - p)} \int_x^1 (1-t)^{\alpha-p} f_p(t) \, dt
\]

for \( 0 < x < 1 \). From (2.4) we have that

\[
(2.5)
\]

\[
R_{\alpha,p}(x) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - p)}(1-x)^{\alpha-p} f_{p+1}(x)
\]

\[
- \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - p)}(\alpha-p) \int_x^1 (1-t)^{\alpha-p-1} f_{p+1}(t) \, dt
\]

\[
= \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - p)}(1-x)^{\alpha-p} f_{p+1}(x) - \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - p)} \int_x^1 (1-t)^{\alpha-p-1} f_{p+1}(t) \, dt
\]

for \( 0 < x < 1 \), where in the last step we have used a standard property of the Gamma function. We now use the induction hypothesis to conclude that

\[
h_{\alpha}(x) = \sum_{n=1}^{p} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 2 - n)} (1-x)^{\alpha+1-n} f_n(x) + R_{\alpha,p}(x)
\]

\[
= \sum_{n=1}^{p+1} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 2 - n)} (1-x)^{\alpha+1-n} f_n(x) + R_{\alpha,p+1}(x)
\]

for \( 0 < x < 1 \), where the last equality follows from (2.5). This proves (2.3) for \( k = p + 1 \). The proof is now completed by the principle of induction. \( \square \)

**Corollary 2.1.** Let \( \alpha \) be a non-negative integer and let \( h_{\alpha} \) be as in (0.1). Then

\[
h_{\alpha}(x) = \sum_{n=1}^{\alpha+1} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 2 - n)} (1-x)^{\alpha+1-n} f_n(x)
\]

for \( 0 < x < 1 \), where the functions \( f_n \) are as in (2.1).

**Proof.** For \( \alpha \) a non-negative integer, the remainder term \( R_{\alpha,k} \) vanishes identically when \( k \geq \alpha + 1 \). The result follows by Theorem 2.1. \( \square \)

We proceed to study the functions \( f_k \) in some more detail. By performing successive partial integrations we obtain

\[
f_1(x) = \log(x)
\]

\[
f_2(x) = x \log(x) + 1 - x
\]

\[
f_3(x) = \frac{x^2}{2} \log(x) + \frac{1}{2} (1-x) - \frac{3}{2} (1-x)^2
\]

\[
f_4(x) = \frac{x^3}{3} \log(x) + \frac{1}{3} (1-x) - \frac{5}{3} (1-x)^2 + \frac{11}{15} (1-x)^3
\]

\[
f_5(x) = \frac{x^4}{4} \log(x) + \frac{1}{4} (1-x) - \frac{7}{4} (1-x)^2 + \frac{11}{12} (1-x)^3 - \frac{25}{28} (1-x)^4
\]

and so on.
Theorem 2.2. Let $P_k$ be as in (2.1) for $k \geq 1$. Then

$$(2.6) \quad f_k(x) = \frac{x^{k-1}}{(k-1)!} \log(x) + P_k(x)$$

for $x > 0$, where $P_k$ is a polynomial of degree at most $k - 1$.

Proof. The proposition holds for $k = 1$ since $f_1(x) = \log(x)$ by construction. Suppose that equation (2.6) is true for some $k \geq 1$. Then

$$f_{k+1}(x) = \int_1^x f_k(t) \, dt = \frac{x^k}{k!} \log(x) + \frac{1 - x^k}{k \cdot k!} + \int_1^x P_k(x) \, dx = \frac{x^k}{k!} \log(x) + P_{k+1}(x),$$

which proves the proposition by induction. \qed

From the above proof we have that $P_1 = 0$ and

$$\begin{cases}
    P_k(x) &= -\frac{x^{k-2}}{(k-1)!} + P_{k-1}(x), \\
    P_k(1) &= 0,
\end{cases}$$

for $k \geq 2$, where the $P_k$’s are as in Proposition 2.1.

Theorem 2.2. Let $f_k$ be as in (2.1) for some $k \geq 1$. Then

$$f_k(x) = \frac{x^{k-1}}{(k-1)!} \log(x) + \frac{1}{(k-1)!} \sum_{j=1}^{k-1} \left( \sum_{m=1}^{j} \frac{(-1)^m}{m} \binom{k-1}{j-m} \right) (x-1)^j$$

for $x > 0$.

Proof. We first consider the leading term of the function $f_k$. Expanding in a Taylor series we have that

$$\frac{x^{k-1}}{(k-1)!} \log(x) = \frac{1}{(k-1)!} \left( 1 + (x-1) \right)^{k-1} \log(1 + (x-1))$$

$$= \frac{1}{(k-1)!} \left( \sum_{j=0}^{k-1} \binom{k-1}{j} (x-1)^j \right) \left( \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j} (x-1)^j \right)$$

$$= \frac{1}{(k-1)!} \sum_{j=0}^{\infty} \sum_{(m_1, m_2) \in A_j} \binom{k-1}{m_1} \frac{(-1)^{m_2+1}}{m_2} (x-1)^j$$

for $|x-1| < 1$, where

$$A_j = \left\{ (m_1, m_2) \in \mathbb{N}^2 : m_1 + m_2 = j, \ 0 \leq m_1 \leq k-1 \text{ and } m_2 \geq 1 \right\}.$$ 

Setting $m = m_2 = j - m_1$ gives that

$$(2.7) \quad \frac{x^{k-1}}{(k-1)!} \log(x) = \frac{1}{(k-1)!} \sum_{j=1}^{\infty} \sum_{m=\max(1,j-k+1)}^{j} \binom{k-1}{j-m} \frac{(-1)^{m+1}}{m} (x-1)^j$$

for $|x-1| < 1$.

From Proposition 2.1 we know that

$$f_k(x) = \frac{x^{k-1}}{(k-1)!} \log(x) + P_k(x),$$

where $P_k$ is a polynomial of degree at most $k - 1$. From Lemma 2.1 we have that

$$f_k(x) = \mathcal{O}((x-1)^k)$$
as $x \to 1$. By a well-known uniqueness property of the Taylor expansion, these two facts together with (2.7) allow us to conclude that

$$P_k(x) = -\frac{1}{(k-1)!} \sum_{j=1}^{k-1} \sum_{m=1}^{j} \frac{(-1)^{m+1}}{m} \left(\frac{k-1}{j-m}\right) (x-1)^j.$$ 

This yields the conclusion of the theorem. □

Recall Taylor’s formula: If $g$ is a function of type $C^n$ in a neighborhood of $a \in \mathbb{R}$, then

$$(2.8) \quad g(x) = \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (x-a)^k + R_n(x),$$

where the remainder $R_n$ is the integral

$$(2.9) \quad R_n(x) = \frac{(x-a)^n}{(n-1)!} \int_0^1 (1-t)^{n-1} g^{(n)}(a+t(x-a)) \, dt.$$ 

See for example [15, Chapter XIII] for more details.

**Theorem 2.3.** Let $f_k$ be as in (2.1) for some $k \geq 1$. Then

$$(2.10) \quad f_k(x) = \frac{(x-1)^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f_k(1+t(x-1)) \, dt$$

for $x > 0$.

**Proof.** Our functions $f_k$ are $C^k$ around 1, and from Lemma 2.1 we know that $f_k^{(j)}(1) = 0$ for $j = 0, \ldots, k-1$. Therefore formula (2.8) together with (2.9) gives that

$$f_k(x) = \frac{(x-1)^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f_k(1+t(x-1)) \, dt$$

for $x > 0$. But $f_k^{(k)}(x) = f_1'(x) = 1/x$, so

$$f_k(x) = \frac{(x-1)^k}{(k-1)!} \int_0^1 (1-t)^{k-1} \frac{1}{1+t(x-1)} \, dt$$

for $x > 0$. □

Let $(a)_n = \Gamma(a+n)/\Gamma(a)$ be the so-called Pochhammer symbol. The hypergeometric function is defined by

$$(2.10) \quad F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n \, n!} z^n, \quad z \in \mathbb{D},$$

for $c \neq 0, -1, -2, \ldots$.

We mention that

$$f_k(x) = \frac{(-1)^k}{k!} (1-x)^k F(1,1;k+1;1-x)$$

for $|x-1| < 1$. In fact, this latter formula follows by Theorem 2.3 and the so-called Euler integral formula for the hypergeometric function, see formula (3.6) below.
3. Formulas for the remainder term

In this section we shall derive two more integral formulas for the remainder term $R_{α,k}$. We proceed to study an associated function $g_α$ appearing in Theorem 3.2 below.

**Theorem 3.1.** Let $α > -1$ and $k ≥ 1$. Let the function $R_{α,k}$ be as in (2.2). Then

$$R_{α,k}(x) = \frac{(-1)^{k+1} \Gamma(α + 1)}{\Gamma(α + 1 - k)(k - 1)!} \int_0^1 (1 - s)^{k-1} \left( \int_x^1 \frac{(1 - t)^α}{1 - s(1 - t)} \, dt \right) ds$$

for $0 < x < 1$.

**Proof.** Recall formula (2.2). By Theorem 2.3 we have that

$$R_{α,k}(x) = \frac{\Gamma(α + 1)}{\Gamma(α + 1 - k)} \int_x^1 (1 - t)^{α-k}(t - 1)^k \left( \int_0^1 \frac{(1 - s)^{k-1}}{1 + s(t - 1)} \, ds \right) \, dt.$$

A change in the order of integration results in

$$R_{α,k}(x) = \frac{(-1)^{k+1} \Gamma(α + 1)}{\Gamma(α + 1 - k)\Gamma(k)} \int_0^1 (1 - s)^{k-1} \left( \int_x^1 \frac{(1 - t)^α}{1 - s(1 - t)} \, dt \right) ds$$

for $0 < x < 1$. □

We are now ready for our main integral formula for the remainder term.

**Theorem 3.2.** Let $α > -1$ and $k ≥ 1$. Let the function $R_{α,k}$ be as in (2.2). Then

$$R_{α,k}(x) = \frac{(-1)^{k+1} \Gamma(α + 1)}{\Gamma(α + 1 - k)(k - 1)!} (1 - x)^{α+1} \int_0^1 (1 - t)^{k-1} g_α((1 - x)t) \, dt$$

for $0 < x < 1$, where

$$g_α(u) = \sum_{n=0}^{∞} \frac{u^n}{α + 1 + n}$$

for $0 < u < 1$.

**Proof.** From Theorem 3.1 we have that

$$R_{α,k}(x) = \frac{(-1)^{k+1} \Gamma(α + 1)}{\Gamma(α + 1 - k)(k - 1)!} \int_0^1 (1 - s)^{k-1} f(s, x) \, ds,$$

where

$$f(s, x) = \int_x^1 \frac{(1 - t)^α}{1 - s(1 - t)} \, dt.$$

A Taylor expansion yields

$$f(s, x) = \sum_{n=0}^{∞} s^n \int_x^1 (1 - t)^{α+n} \, dt = \sum_{n=0}^{∞} s^n \frac{(1 - x)^{α+1+n}}{α + 1 + n} = (1 - x)^{α+1} g_α(s(1 - x))$$

for $0 < x < 1$ and $0 < s < 1$, which gives the desired formula. □

Let us comment on the function $g_α$ appearing in Theorem 3.2. Observe that the function $g_α$ is the restriction to the unit interval of the function

$$g_α(z) = \sum_{n=0}^{∞} \frac{z^n}{α + 1 + n}, \quad z ∈ ℂ,$$

analytic in the open unit disc $ℂ$. 
Lemma 3.1. Let $\alpha > -1$ and consider the function $g_{\alpha}$ given by (3.2). Then the function $g_{\alpha}$ continues to an analytic function in the slit plane $\mathbb{C} \setminus [1, \infty)$. Furthermore, \( g_{\alpha}(z) = \int_0^1 \frac{t^\alpha}{1 - tz} \, dt \) for $z \in \mathbb{C} \setminus [1, \infty)$.

Proof. A Taylor expansion gives
\[
\int_0^1 \frac{t^\alpha}{1 - tz} \, dt = \sum_{n \geq 0} z^n \int_0^1 t^{\alpha+n} \, dt = \sum_{n \geq 0} \frac{z^n}{\alpha + 1 + n} = g_{\alpha}(z)
\]
for $z \in D$. The lemma now follows by a standard analytic continuation argument. \(\square\)

A calculation shows that
\[
g_{0}(z) = -\frac{\log(1 - z)}{z}
\]
for $z \in \mathbb{C} \setminus [1, \infty)$, where log is the principal branch logarithm.

A differentiation shows that
\[
g_{\alpha}^{(j)}(z) = j! \int_0^1 \frac{t^{\alpha+j}}{(1 - tz)^{j+1}} \, dt
\]
for $z \in \mathbb{C} \setminus [1, \infty)$ and $j \geq 0$. Thus the function $g_{\alpha}$ is totally monotone on $(-\infty, 1)$ in the sense that $g_{\alpha}^{(j)}(x) \geq 0$ for $x < 1$ and $j \geq 0$.

We observe that
\[
g_{\alpha}^{(j)}(z) = \frac{j!}{\alpha + j + 1} F\left(j + 1, \alpha + j + 1; \alpha + j + 2; z\right),
\]
where $F$ is the hypergeometric function, see (2.10). In fact, formula (3.4) is a specialization of the Euler integral formula
\[
F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} \, dt
\]
valid when $c > b > 0$ (see [3, Theorem 2.2.1]).

The study of asymptotic behavior of hypergeometric functions is a classical topic which goes back to Gauss. The principal result that \(\lim_{x \to 1} F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)}\Gamma(c - a - b)\Gamma(c - a)\Gamma(c - b)\) valid when $c > a + b$ is known as Gauss summation formula (see [3, Theorem 2.2.2]). The case $c < a + b$ can be reduced to (3.7) by means of the transformation formula
\[
F(a, b; c; x) = (1 - x)^{c-a-b}F(c - a, c - b; c; x)
\]
known as Euler’s formula (see [3, Theorem 2.2.5]). We shall use also the transformation formula
\[
F(a, b; c; x) = (1 - x)^{-a}F(a, c - b; c; \frac{x}{x - 1})
\]
known as Pfaff’s formula (see [3, Theorem 2.2.5]).

In our case we have that
\[
\lim_{x \to 1} g_{\alpha}(x) = 1 \quad \text{and} \quad \lim_{x \to 1} g_{\alpha}^{(j)}(x)(1 - x)^j = (j - 1)!
\]
for $\alpha > -1$ and $j \geq 1$ (see [3, Theorem 2.1.3]).

We now turn to asymptotics of $g^{(j)}_{\alpha}(x)$ as $x \to -\infty$.

Lemma 3.2. Let $\alpha > 0$. Then

$$
\lim_{x \to -\infty} g^{(j)}_{\alpha}(x)(1 - x)^{j+1} = j! / \alpha
$$

for $j \geq 0$.

Proof. Recall formula (3.5). By Pfaff's transformation formula (3.9) we have that

$$
g^{(j)}_{\alpha}(x) = \frac{j!}{\alpha + 1 + j} (1 - x)^{-(j+1)} F(j + 1, 1; \alpha + 2 + j; \frac{x}{x - 1}).
$$

Notice that $x/(x-1) \to 1$ as $x \to -\infty$. An application of Gauss summation formula (3.7) yields that

$$
\lim_{x \to -\infty} g^{(j)}_{\alpha}(x)(1 - x)^{j+1} = \frac{j!}{\alpha + 1 + j} \frac{\Gamma(\alpha + 2 + j)\Gamma(\alpha)}{\Gamma(\alpha + 1)\Gamma(\alpha + 1 + j)} = \frac{j!}{\alpha},
$$

where the last equality is straightforward to check. \qed

We now consider the case $-1 < \alpha < 0$.

Lemma 3.3. Let $-1 < \alpha < 0$. Then

$$
\lim_{x \to -\infty} g^{(j)}_{\alpha}(x)(1 - x)^{\alpha+1+j} = \frac{\Gamma(\alpha + 1 + j)}{\Gamma(\alpha + 1)} \frac{\pi}{\sin((\alpha + 1)\pi)}
$$

for $j \geq 0$.

Proof. We proceed as in the proof of Lemma 3.2. Recall formula (3.11). By Euler's transformation formula (3.8) we have that

$$
g^{(j)}_{\alpha}(x) = \frac{j!}{\alpha + 1 + j} (1 - x)^{-\alpha} F(\alpha + 1, \alpha + 1 + j; \alpha + 2 + j; \frac{x}{x - 1})
$$

$$
= \frac{j!}{\alpha + 1 + j} (1 - x)^{-(\alpha+1)} F(\alpha + 1, \alpha + 1 + j; \alpha + 2 + j; \frac{x}{x - 1}),
$$

where the last equality is straightforward to check. Notice that $x/(x-1) \to 1$ as $x \to -\infty$. An application of Gauss summation formula (3.7) yields that

$$
\lim_{x \to -\infty} g^{(j)}_{\alpha}(x)(1 - x)^{\alpha+1+j} = \frac{j!}{\alpha + 1 + j} \frac{\Gamma(\alpha + 2 + j)\Gamma(-\alpha)}{\Gamma(j + 1)\Gamma(1)} = \Gamma(\alpha + 1 + j)\Gamma(-\alpha),
$$

where the last equality is straightforward to check. We now use Euler's reflection formula (1.2) to arrive at the conclusion of the lemma. \qed

Using formula (3.3) it is straightforward to check that

$$
g^{(j)}_{\alpha}(x) = (-1)^{j+1} j! \log(1 - x) \frac{1}{x^{j+1}} + O(1/x^{j+1})
$$

as $x \to -\infty$ for $j \geq 0$. 

4. The relative remainder term

In this section we introduce the relative remainder term and begin our study of its limit behavior.

Let $\alpha > -1$. We shall study the relative remainder term defined by

\begin{equation}
E_{\alpha,k}(z) = \frac{1}{k} \int_0^1 (1-t)^{k-1} g_{\alpha}((1-z)t) \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0],
\end{equation}

for $k = 1, 2, \ldots$. We have that $E_{\alpha,k}$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$ since by Lemma 3.1 the function $g_{\alpha}$ is analytic in $\mathbb{C} \setminus [1, \infty)$.

From the integral formula in Theorem 3.2 we have that

\begin{equation}
R_{\alpha,k}(x) = c_{\alpha,k}(1-x)^{\alpha+1} E_{\alpha,k}(x)
\end{equation}

for $0 < x < 1$, where

$$ c_{\alpha,k} = \frac{(-1)^{k+1} \Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - k) k!}. $$

Using the reflection formula (1.2) it is straightforward to check that this constant $c_{\alpha,k}$ has the alternative form

$$ c_{\alpha,k} = \frac{\sin(\alpha \pi) \Gamma(\alpha + 1) \Gamma(k - \alpha)}{\pi k!}. $$

From Stirling’s formula we have that

\begin{equation}
\lim_{k \to +\infty} k^{\alpha+1} c_{\alpha,k} = \frac{\sin(\alpha \pi) \Gamma(\alpha + 1)}{\pi}
\end{equation}

(see [3, Section 1.4]).

Recall that

$$ B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt, \quad a, b > 0, $$

is the Beta function. It is well-known that $B(a,b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ for $a, b > 0$, where $\Gamma$ is the Gamma function (see for instance [1, formula (6.2.2)] or [3, Section 1.1]).

We set

\begin{equation}
f(t; a, b) = \frac{1}{B(a,b)} t^{a-1} (1-t)^{b-1}, \quad 0 < t < 1,
\end{equation}

for $a, b > 0$. Notice that the function $f(\cdot; a, b)$ is non-negative and has the property that $\int_0^1 f(t; a, b) \, dt = 1$ for $a, b > 0$.

**Lemma 4.1.** Let $a > 0$. Let $\{\varphi_r\}_{r \in I}$ be a family of measurable functions on $(0,1)$ such that

$$ C = \sup_{r \in I} \int_0^1 |\varphi_r(t)| (1-t)^N \, dt < +\infty $$

for some $N \geq 0$. Assume that $\varphi_r(0) = \lim_{t \to 0} \varphi_r(t)$ uniformly for $r \in I$. Then

$$ \lim_{b \to +\infty} \frac{1}{B(a,b)} \int_0^1 t^{a-1} (1-t)^{b-1} \varphi_r(t) \, dt = \varphi_r(0) $$

uniformly for $r \in I$. 

Proof. We first show that

(4.5) \[ C_1 = \sup_{r \in I} |\varphi_r(0)| < +\infty. \]

Since \( \varphi_r(0) = \lim_{t \to 0} \varphi_r(t) \) uniformly for \( r \in I \), there exists \( 0 < \delta_1 < 1 \) such that \( |\varphi_r(0)| \leq |\varphi_r(t)| + 1 \) for \( r \in I \) if \( 0 < t < \delta_1 \). An integration now shows that

\[
|\varphi_r(0)| \int_0^{\delta_1} (1 - t)^N dt \leq \int_0^{\delta_1} |\varphi_r(t)|(1 - t)^N dt + \int_0^{\delta_1} (1 - t)^N dt.
\]

Thus \( |\varphi_r(0)| \leq C/C_2 + 1 \), where \( C_2 = \int_0^{\delta_1} (1 - t)^N dt \) and \( C > 0 \) is as in the lemma. This proves (4.5).

The lemma now follows by a summability kernel argument. Let \( f(;a,b) \) be as in (4.4). Let \( \varepsilon > 0 \) be given. Since \( \varphi_r(0) = \lim_{t \to 0} \varphi_r(t) \) uniformly for \( r \in I \), there exists \( 0 < \delta < 1 \) such that \( |\varphi_r(t) - \varphi_r(0)| < \varepsilon/3 \) for \( r \in I \) if \( 0 < t < \delta \). Observe that

\[
\int_0^1 f(t; a, b) |\varphi_r(t)| dt - |\varphi_r(0)| = \int_0^1 f(t; a, b) (|\varphi_r(t)| - |\varphi_r(0)|) dt.
\]

From the triangle inequality we have that

(4.6) \[ \left| \int_0^1 f(t; a, b) |\varphi_r(t)| dt - |\varphi_r(0)| \right| \leq \left( \int_0^\delta + \int_\delta^1 \right) f(t; a, b) |\varphi_r(t) - \varphi_r(0)| dt \]

\[ \leq \varepsilon/3 + \int_\delta^1 f(t; a, b) |\varphi_r(t)| dt + \left( \int_\delta^1 f(t; a, b) dt \right) |\varphi_r(0)| \]

for \( r \in I \) and \( b > 0 \).

We next estimate the mid term on the right hand side in (4.6). By a standard estimation we have that

\[
\int_\delta^1 f(t; a, b) |\varphi_r(t)| dt \leq \frac{1}{B(a, b)} \left( \sup_{b < t < 1} t^{a-1} (1 - t)^{b-N-1} \right) \int_\delta^1 |\varphi_r(t)|(1 - t)^N dt \]

\[ \leq \frac{C}{B(a, b)} \cdot \frac{(1 - \delta)^{b-N-1}}{\delta} \]

for \( b > N + 1 \), where \( C > 0 \) is as in the lemma. From Stirling’s formula we have that \( 1/B(a, b) \asymp b^a / \Gamma(a) \) as \( b \to \infty \) in the sense that the quotient tends to 1. By geometric decay there exists \( b_1 > 0 \) such that

\[
\int_\delta^1 f(t; a, b) |\varphi_r(t)| dt < \varepsilon/3
\]

for \( r \in I \) if \( b > b_1 \).

We next consider the rightmost term in (4.6). Similarly as in the previous paragraph we have that \( \lim_{t \to \infty} \int_\delta^1 f(t; a, b) dt = 0 \). We choose \( b_2 > b_1 \) such that \( \int_\delta^1 f(t; a, b) dt < \varepsilon/(3C_1) \), where \( C_1 > 0 \) is as in (4.5). By (4.6) we have that

\[
\left| \int_0^1 f(t; a, b) |\varphi_r(t)| dt - |\varphi_r(0)| \right| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\]

for \( r \in I \) if \( b > b_2 \). This completes the proof of the lemma. \( \square \)

We next specialize Lemma 4.1.
Lemma 4.2. Let $a > 0$. Let $\varphi$ be a measurable function on $(-A, 1)$ for some real number $A > 1$ such that

$$
\int_{-A}^{0} |\varphi(t)| \, dt + \int_{0}^{1} |\varphi(t)| (1 - t)^N \, dt < +\infty
$$

for some $N \geq 0$. If $\lim_{t \to 0} \varphi(t) = \varphi(0)$, then

$$
\lim_{b \to +\infty} \frac{1}{B(a, b)} \int_{0}^{1} t^{n-1}(1-t)^{b-1} \varphi(rt) \, dt = \varphi(0)
$$

uniformly for $-A < r < 1$.

Proof. We shall apply Lemma 4.1. Set

$$
\varphi_r(t) = \varphi(rt), \quad 0 < t < 1,
$$

for $-A < r < 1$. Clearly $\varphi(0) = \lim_{t \to 0} \varphi_r(t)$ uniformly for $-A < r < 1$.

It remains to check some appropriate $L^1$-bounds for the family $\{\varphi_r\}$. Since $\lim_{t \to 0} \varphi(t) = \varphi(0)$, there exists $0 < \delta < 1$ such that $|\varphi(t)| \leq |\varphi(0)| + 1$ for $|t| < \delta$. Therefore $\int_{0}^{1} |\varphi_r(t)| \, dt \leq |\varphi(0)| + 1$ for $|r| < \delta$.

We next consider the case $\delta < r < 1$. By a change of variables we have that

$$
\int_{0}^{1} |\varphi_r(t)| (1 - t)^N \, dt = \frac{1}{r} \int_{0}^{1} |\varphi(t)|(1 - t/r)^N \, dt \leq \frac{1}{\delta} \int_{0}^{1} |\varphi(t)| (1 - t)^N \, dt
$$

for such $r$'s, where the last equality follows by a monotonicity argument.

We consider the case $-\delta < r < -A$. By a change of variables we have that

$$
\int_{0}^{1} |\varphi_r(t)| \, dt = \frac{1}{|r|} \int_{r}^{1} |\varphi(t)| \, dt \leq \frac{1}{\delta} \int_{-A}^{0} |\varphi(t)| \, dt
$$

for such $r$'s. This completes the proof of the lemma.

We now return to the relative remainder term $E_{\alpha, k}$.

Theorem 4.1. Let $\alpha > -1$ and $j \geq 0$. Let the function $E_{\alpha, k}$ be as in (4.1). Then

$$
\lim_{k \to +\infty} k^j E_{\alpha, k}^{(j)}(x) = \frac{(-1)^j (j)!^2}{\alpha + 1 + j}
$$

uniformly for $0 < x < A$ for every real number $A > 0$.

Proof. Recall formula (4.1). Differentiating we see that

$$
E_{\alpha, k}^{(j)}(x) = (-1)^j k \int_{0}^{1} t^j (1 - t)^{k-1} g_{\alpha}^{(j)}((1 - x)t) \, dt
$$

for $x > 0$. Observe that

$$
B(j + 1, k) = \frac{\Gamma(j + 1) \Gamma(k)}{\Gamma(j + k + 1)} = \frac{j!(k-1)!}{(k+j)!},
$$

which inserted in formula (4.7) gives

$$
E_{\alpha, k}^{(j)}(x) = (-1)^j \frac{j!k!}{(k+j)!} \frac{1}{B(j + 1, k)} \int_{0}^{1} t^j (1 - t)^{k-1} g_{\alpha}^{(j)}((1 - x)t) \, dt.
$$
It is straightforward to check that \( k!/(k+j)! \approx 1/k^j \) as \( k \to \infty \) in the sense that the quotient tends to 1. From (3.10) we have that the function \( \varphi = g^{(j)}_\alpha \) satisfies the growth assumption of Lemma 4.2. By Lemma 4.2 it follows that

\[
\lim_{k \to \infty} k^j E^{(j)}_{\alpha,k}(x) = (-1)^j j! g^{(j)}_\alpha(0)
\]

uniformly for \( 0 < x < A+1 \). From formula (3.2) we have that \( g^{(j)}(0) = j!/(\alpha+1+j) \). We conclude that

\[
\lim_{k \to \infty} k^j E^{(j)}_{\alpha,k}(x) = \frac{(-1)^j (j!^2)}{\alpha + 1 + j}
\]

uniformly for \( 0 < x < A + 1 \). \( \square \)

Let us go back to the remainder term \( R_{\alpha,k} \).

**Corollary 4.1.** Let \( \alpha > -1 \). Let the function \( R_{\alpha,k} \) be as in (2.2). Then

\[
\lim_{k \to +\infty} k^{\alpha+1} R_{\alpha,k}(x) = \frac{\sin(\alpha \pi)}{\pi} \frac{\Gamma(\alpha+1)}{(1-x)^{\alpha+1}}
\]

uniformly for \( 0 < x < 1 \).

**Proof.** Recall formula (4.2). We also have (4.3). The result now follows by Theorem 4.1. \( \square \)

5. **Calculation of the scaled limit**

The purpose of this section is to calculate the scaled limit of the relative remainder term. Let us begin with some statistical considerations.

A random variable \( X : \Omega \to (0,1) \) is said to be Beta\((a, b)\)-distributed with parameters \( a > 0 \) and \( b > 0 \) if its probability density function has the form \( f(\cdot; a, b) \), where \( f(\cdot; a, b) \) is as in (4.4). It is well-known that the expectation and variance of such a random variable \( X \) are

\[
E[X] = \frac{a}{a + b} \quad \text{and} \quad \text{Var}[X] = \frac{ab}{(a + b)(a + b + 1)},
\]

respectively (see [8, Chapter 5]).

We denote by \( C_0[0, \infty) \) the space of continuous functions \( \varphi \) on the closed half-axis \([0, \infty)\) such that \( \lim_{t \to \infty} \varphi(t) = 0 \). The space \( C_0[0, \infty) \) is equipped with usual maximum norm \( \| \varphi \|_\infty = \sup_{t \geq 0} |\varphi(t)| \).

We shall take as our starting point the following well-known lemma. For the sake of completeness of discussion we include some details of proof.

**Lemma 5.1.** Let \( a > 0 \). Then

\[
\lim_{b \to +\infty} \frac{1}{B(a,b)} \int_0^1 t^{a-1}(1-t)^{b-1} \varphi(bt) \, dt = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}e^{-t} \varphi(t) \, dt
\]

for \( \varphi \in C_0[0, \infty) \).

**Proof.** Let \( \varphi \in C_0[0, \infty) \). A change of variables gives that

\[
\frac{1}{B(a,b)} \int_0^1 t^{a-1}(1-t)^{b-1} \varphi(bt) \, dt = \frac{b^{-a}}{B(a,b)} \int_0^b t^{a-1} \left(\frac{1-t}{b}\right)^{b-1} \varphi(t) \, dt.
\]

From Stirling’s formula we have that

\[
\frac{b^{-a}}{B(a,b)} \Gamma(a) \Gamma(b) b^{-a} \to \frac{1}{\Gamma(a)}
\]
as \( b \to +\infty \) (see [3, Section 1.4]).

Recall that the logarithm is concave on \((0, \infty)\). This concavity yields that
\[
\log(x) \leq x - 1 \quad \text{for} \quad x > 0.
\]
As a consequence we have that
\[
\left(1 - \frac{t}{b}\right)^{b-1} = e^{(b-1) \log(1-t/b)} \leq e^{-t/2}
\]
for \( b \geq 2 \) and \( 0 < t < b \). Since \( \lim_{b \to \infty} (1 - t/b)^{b-1} = e^{-t} \) we obtain from the dominated convergence theorem that
\[
\lim_{b \to +\infty} \frac{1}{B(a, b)} \int_0^1 t^{a-1}(1-t)^{b-1} \varphi(bt) \, dt = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-t} \varphi(t) \, dt
\]
for \( \varphi \in C_0(0, \infty) \).

Lemma 5.1 has a certain statistical meaning. Let \( X_{a,b} \) be a Beta\((a, b)\)-distributed random variable. Lemma 5.1 says that the random variable \( bX_{a,b} \) has limit \( Y \) in distribution as \( b \to +\infty \), where the random variable \( Y \) is Gamma\((a, 1)\)-distributed.

Let us return to the space \( C_0(0, \infty) \) of continuous functions on \([0, \infty)\) vanishing at infinity. A classical result identifies the dual of \( C_0(0, \infty) \) as the space \( M(0, \infty) \) of complex regular Borel measures on \([0, \infty)\) equipped with the norm of finite total variation, see for instance Rudin [21, Chapter 6] for details. Following usual practice we say that a sequence \( \{\Lambda_k\}_{k=1}^\infty \) in \( M(0, \infty) \) converges weak* to \( \Lambda_0 \) in \( M(0, \infty) \) if \( \lim_{k \to \infty} \Lambda_k(\varphi) = \Lambda_0(\varphi) \) for every \( \varphi \in C_0(0, \infty) \). Notice that Lemma 5.1 has a natural interpretation in terms of weak* convergence in \( M(0, \infty) \).

The space \( C_0(0, \infty) \) has a natural structure of a homogeneous Banach space on the multiplicative group of positive real numbers. For \( r > 0 \) we define
\[
T_r \varphi(t) = \varphi(t/r), \quad t \geq 0,
\]
for \( \varphi \in C_0(0, \infty) \). Notice that every \( T_r \) is a linear operator on \( C_0(0, \infty) \) such that \( \|T_r \varphi\| = \|\varphi\| \) for \( \varphi \in C_0(0, \infty) \). Observe that
\[
T_r T_s = T_{rs}, \quad r, s > 0,
\]
and that \( T_1 = I \) is the identity operator on \( C_0(0, \infty) \). The group of translation operators
\[
(0, \infty) \ni r \mapsto T_r \in \mathcal{L}(C_0(0, \infty))
\]
is continuous in the strong operator topology (SOT) in \( \mathcal{L}(C_0(0, \infty)) \). In view of the group property (5.1) this latter SOT continuity boils down to the fact that \( \lim_{r \to 1} T_r \varphi = \varphi \) in \( C_0(0, \infty) \) for every \( \varphi \in C_0(0, \infty) \). We refer to the classical book Katznelson [13] for further background.

**Theorem 5.1.** Let \( \alpha > -1 \) and \( j \geq 0 \). Let \( E_{\alpha, k} \) be as in (4.1). Then
\[
\lim_{k \to +\infty} \frac{k^j}{j!} E_{\alpha, k}^{(j)}(kx) = \int_0^\infty (-t)^j e^{-t} g_\alpha^{(j)}(-xt) \, dt
\]
uniformly for \( x \in K \) for every compact subset \( K \) of \((0, \infty)\).

**Proof.** From formula (4.1) we have that
\[
E_{\alpha, k}^{(j)}(x) = (-1)^j k \int_0^1 t^j (1-t)^{k-1} g_\alpha^{(j)}((1-x)t) \, dt
\]
\[
= (-1)^j \frac{j!k!}{(k+j)!} B(j+1, k) \int_0^1 t^j (1-t)^{k-1} g_\alpha^{(j)}((1-x)t) \, dt
\]
for $x > 0$. Furthermore
\[
\frac{k!}{(k+j)!} \leq \frac{1}{k^j}
\]
as $k \to +\infty$ in the sense that the quotient tends to 1 as $k \to +\infty$. To complete the proof of the theorem it thus suffices to show that
\[
\lim_{k \to +\infty} \frac{1}{B(j+1,k)} \int_0^1 t^j (1-t)^{k-1} g_k(\alpha)(1-kx)t \, dt = \frac{1}{j!} \int_0^\infty t^j e^{-t} g_k^{(j)}(-xt) \, dt
\]
uniformly for $x \in K$ for every compact subset $K$ of $(0, \infty)$.

We introduce the linear functionals
\[
\Lambda_k(\varphi) = \frac{1}{B(j+1,k)} \int_0^1 t^j (1-t)^{k-1} \varphi(kt) \, dt, \quad \varphi \in C_0[0, \infty),
\]
for $k = 1, 2, \ldots$ and
\[
\Lambda_0(\varphi) = \frac{1}{j!} \int_0^\infty t^j e^{-t} \varphi(t) \, dt, \quad \varphi \in C_0[0, \infty),
\]
on the space $C_0[0, \infty)$. Notice that the functional $\Lambda_k$ is induced by a probability measure on $[0, \infty) (k = 0, 1, 2, \ldots)$. From Lemma 5.1 we know that $\Lambda_k \to \Lambda_0$ weak* in $M[0, \infty)$ as $k \to \infty$. As a consequence we have that
\[
\lim_{k \to \infty} \Lambda_k(\varphi) = \Lambda_0(\varphi)
\]
uniformly for $\varphi \in K$ for every compact subset $K$ of $C_0[0, \infty)$, see for instance [13, Remark I.2.8]. We shall apply the uniform limit (5.2) to a naturally derived compact set $\mathcal{K}$ of test functions.

Set
\[
\varphi(t) = g_k^{(j)}(-t), \quad t \geq 0.
\]
From Lemma 3.2, Lemma 3.3 and (3.12) we know that $\varphi \in C_0[0, \infty)$. Observe that
\[
T_{1/x} T_{kx/(kx-1)} \varphi(t) = g_k^{(j)}((1-kx)/t/k), \quad t \geq 0,
\]
when $x > 0$ and $kx > 1$. Furthermore,
\[
\frac{1}{B(j+1,k)} \int_0^1 t^j (1-t)^{k-1} g_k^{(j)}((1-kx)t) \, dt = \Lambda_k(T_{1/x} T_{kx/(kx-1)} \varphi)
\]
when $x > 0$ and $kx > 1$. Notice also that
\[
\frac{1}{j!} \int_0^\infty t^j e^{-t} g_k^{(j)}(-xt) \, dt = \Lambda_0(T_{1/x} \varphi)
\]
for $x > 0$.

Let $\varepsilon > 0$ be given. Let $K$ be a compact subset of $(0, \infty)$, say, $K \subset [1/A, A]$ for some $A > 1$. Since $\lim_{r \to 1} T_{r} \varphi = \varphi$ in $C_0[0, \infty)$, there exists $\delta_1 > 0$ such that $\|T_{r} \varphi - \varphi\|_{\infty} < \varepsilon/2$ if $|r - 1| < \delta_1$. Notice that
\[
|kx/(kx-1) - 1| = 1/(kx - 1) < \delta_1
\]
if $kx > 1/\delta_1 + 1$. Choose $k_1 > (1/\delta_1 + 1)A$. Now
\[
\|T_{kx/(kx-1)} \varphi - \varphi\|_{\infty} < \varepsilon/2
\]
for all $x \in K$ if $k > k_1$. We shall next apply (5.2) to the compact subset
\[
\mathcal{K} = \{T_{1/x} \varphi : x \in K\}
\]
of \( C_0[0, \infty) \). By such an application there exists \( k_2 > k_1 \) such that

\[
(5.4) \quad |\Lambda_k(T_{1/x} \varphi) - \Lambda_0(T_{1/x} \varphi)| < \varepsilon/2
\]

for all \( x \in K \) if \( k > k_2 \). By the triangle inequality we now have that

\[
\left| \frac{1}{B(j+1, k)} \int_0^1 t^j(1-t)^{k-1} g_{\alpha}^{(j)}((1-kx)t) \, dt - \frac{1}{j!} \int_0^\infty t^j e^{-t} g_{\alpha}^{(j)}(-xt) \, dt \right|
\]

\[
= |\Lambda_k(T_{1/x} T_{kx/(kx-1)} \varphi) - \Lambda_0(T_{1/x} \varphi)|
\]

\[
\leq |\Lambda_k(T_{1/x} T_{kx/(kx-1)} \varphi - \varphi)| + |\Lambda_k(T_{1/x} \varphi) - \Lambda_0(T_{1/x} \varphi)| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

for all \( x \in K \) if \( k > k_2 \), where the strict inequality follows by (5.3) and (5.4). This completes the proof of the theorem. \( \square \)

Observe that the result of Theorem 5.1 restates as

\[
\lim_{k \to +\infty} E_{\alpha, k}(kx) = \int_0^\infty e^{-t} g_{\alpha}(-xt) \, dt
\]

for \( x > 0 \) with convergence in the sense of the space \( C^\infty(0, \infty) \) of \( C^\infty \)-smooth functions in \( (0, \infty) \).

We shall next discuss convergence in the sense of analytic functions. We denote by \( H(\Omega) \) the space of analytic functions in a planar open set \( \Omega \). The space \( H(\Omega) \) is topologized by means of uniform convergence on compact subsets of \( \Omega \), that is, \( f_k \to f_0 \) in \( H(\Omega) \) means that \( f_k \to f_0 \) uniformly on compact subsets of \( \Omega \). Convergence in \( H(\Omega) \) is often referred to as normal convergence in \( \Omega \). It is well-known that the space \( H(\Omega) \) is continuously embedded into the space \( C^\infty(\Omega) \) of \( C^\infty \)-smooth functions in \( \Omega \).

We shall need some more estimates of the function \( g_{\alpha} \). Recall from Lemma 3.1 that

\[
g_{\alpha}(z) = \int_0^1 \frac{t^\alpha}{1-tz} \, dt
\]

for \( z \in \mathbb{C} \setminus [1, \infty) \).

**Lemma 5.2.** Let \( \alpha > -1 \) and let \( g_{\alpha} \) be as above. Then

\[
|g_{\alpha}(z)| \leq \frac{1}{(\alpha + 1) \, |\text{Im}(z)|}
\]

for \( z \in \mathbb{C} \setminus \mathbb{R} \).

**Proof.** Let \( z = x + iy \) with \( x, y \in \mathbb{R} \) and \( y \neq 0 \). A completion of squares gives that

\[
|1 - tz|^2 = (x^2 + y^2) \left( t - \frac{x}{x^2 + y^2} \right)^2 + \frac{y^2}{x^2 + y^2}.
\]

Hence \( |1 - tz|^2 \geq (\text{Im}(z))^2/|z|^2 \) for \( t \in \mathbb{R} \) and \( z \neq 0 \). By the triangle inequality we have that

\[
|g_{\alpha}(z)| \leq \frac{|z|}{|\text{Im}(z)|} \int_0^1 t^\alpha \, dt = \frac{1}{(\alpha + 1) \, |\text{Im}(z)|}
\]

for \( z \in \mathbb{C} \setminus \mathbb{R} \). \( \square \)

**Lemma 5.3.** Let \( \alpha > -1 \) and let \( g_{\alpha} \) be as above. Then \( |g_{\alpha}(z)| \leq g_{\alpha}(x) \) when \( x = \text{Re}(z) < 1 \).
Proof. Let \( z = x + iy \) with \( x < 1 \) and \( y \in \mathbb{R} \). Recall Lemma 3.1. Observe that \( |1 - tz| \geq 1 - tx > 0 \) for \( 0 < t < 1 \). From the triangle inequality we have that

\[
|g_\alpha(z)| \leq \int_0^1 \frac{t^n}{|1 - tz|} \, dt \leq \int_0^1 \frac{t^n}{1 - tx} \, dt = g_\alpha(x),
\]

where the last equality again follows by Lemma 3.1. This completes the proof of \( x = \text{Re}(z) \).

Recall that a subset \( \mathcal{F} \) of \( H(\Omega) \) is called a normal family if every sequence from \( \mathcal{F} \) has a subsequence which converges in \( H(\Omega) \). The limit function is not required to belong to \( \mathcal{F} \).

**Lemma 5.4.** Let \( \alpha > -1 \) and consider the functions

\[
f_k(z) = E_{\alpha,k}(kz) = k \int_0^1 (1 - t)^{k-1} g_\alpha((1 - k)z) \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0],
\]

for \( k = 1, 2, \ldots \). Then the set \( \mathcal{F} = \{ f_k : k = 1, 2, \ldots \} \) is a normal family of analytic functions in the slit plane \( \mathbb{C} \setminus (-\infty, 0] \).

**Proof.** Clearly, the function \( f_k \) is analytic in \( \mathbb{C} \setminus (-\infty, 0] \) \( (k = 1, 2, \ldots) \). We shall show that the functions in the set \( \mathcal{F} \) are uniformly bounded on compact subsets of \( \mathbb{C} \setminus (-\infty, 0] \). By Montel’s theorem it then follows that \( \mathcal{F} \) is a normal family in \( H(\mathbb{C} \setminus (-\infty, 0]) \) (see [21, Theorem 14.6]). We now proceed to details.

Let \( K \) be a compact subset of \( \mathbb{C} \setminus (-\infty, 0] \). By compactness of \( K \) there exists a constant \( C > 0 \) such that \( |z| \leq C \) for \( z \in K \). Since \( 0 \not\in K \), there exists \( 0 < \eta < 1 \) such that \( K \subset \mathbb{C} \setminus (-\infty, \eta] \). Consider now the sets

\[
K_1 = \{ z \in K : \text{Re}(z) \leq \eta \} \quad \text{and} \quad K_2 = \{ z \in K : \text{Re}(z) \geq \eta \}.
\]

Clearly, the set \( K_j \) is compact \( (j = 1, 2) \) and \( K = K_1 \cup K_2 \). By construction there exists \( \delta > 0 \) such that \( \text{Im}(|z|) \geq \delta \) for \( z \in K_1 \).

We consider now the case when \( z \in K_1 \). Observe that

\[
\text{Im}((1 - k)z) = -kt \text{Im}(z) \neq 0
\]

for \( z \in K_1, 0 < t \leq 1 \) and \( k = 1, 2, \ldots \). From Lemma 5.2 we have that

\[
|g_\alpha((1 - k)z)| \leq \frac{1}{\alpha + 1} \frac{|1 - kz|}{|\text{Im}(z)|k} \leq \frac{1}{\alpha + 1} \frac{1 + kC}{\delta k} \leq \frac{1}{\alpha + 1} \frac{1 + C}{\delta}
\]

for \( z \in K_1, 0 < t \leq 1 \) and \( k = 1, 2, \ldots \). From the triangle inequality we now have that

\[
|f_k(z)| \leq \frac{1}{\alpha + 1} \frac{1 + C}{\delta} \left( k \int_0^1 (1 - t)^{k-1} \, dt \right) = \frac{1}{\alpha + 1} \frac{C + 1}{\delta}
\]

for \( z \in K_1 \) and \( k = 1, 2, \ldots \).

We consider now the case when \( z \in K_2 \). Let \( 0 \leq t \leq 1 \), \( z \in K_2 \) and write \( x = \text{Re}(z) \). Observe that

\[
\text{Re}((1 - k)z) = (1 - kx)t \leq 1 - \eta
\]

for \( k = 1, 2, \ldots \). By Lemma 5.3 we have that

\[
|g_\alpha((1 - k)z)| \leq g_\alpha((1 - kx)t)
\]

for \( k = 1, 2, \ldots \). Moreover, since the function \( g_\alpha \) is increasing on \( (-\infty, 1) \), we conclude that

\[
|g_\alpha((1 - k)z)| \leq g_\alpha(1 - \eta)
\]
for \( z \in K_2, 0 \leq t \leq 1 \) and \( k = 1, 2, \ldots \) (see the paragraph containing formula (3.4)). From the triangle inequality we now have that

\[
|f_k(z)| \leq g_a(1 - \eta) \left(1 + t \right)^{k-1} dt = g_a(1 - \eta)
\]

for \( z \in K_2 \) and \( k = 1, 2, \ldots \). This completes the proof of the lemma.

We can now extend Theorem 5.1.

**Theorem 5.2.** Let \( \alpha > -1 \). Let \( E_{a,k} \) be as in (4.1). Then

\[
\lim_{k \to +\infty} E_{a,k}(z) = \int_0^\infty e^{-t} g_a(-zt) dt
\]

uniformly for \( z \in K \) for every compact subset \( K \) of \( \mathbb{C} \setminus (\infty, 0] \).

**Proof.** Set \( \Omega = \mathbb{C} \setminus (-\infty, 0] \). Recall formula (4.1). Set

\[
f_k(z) = E_{a,k}(z), \quad z \in \Omega,
\]

for \( k = 1, 2, \ldots \) and

\[
f_0(z) = \int_0^\infty e^{-t} g_a(-zt) dt, \quad z \in \Omega.
\]

Clearly, the function \( f_k \) is analytic in \( \Omega \) \((k = 0, 1, 2, \ldots)\). By Lemma 5.4 we have that \( \{f_k : k = 1, 2, \ldots\} \) is a normal family of analytic functions in \( \Omega \). By Theorem 5.1 we know that \( \lim_{k \to \infty} f_k(x) = f_0(x) \) for \( x > 0 \).

We now show that \( f_k \to f_0 \) as \( k \to \infty \) in the topology of normal convergence in \( \Omega \). Assume to reach a contradiction that there exists a compact set \( K \subset \Omega \) such that \( \{f_k\}_{k=1}^\infty \) does not converge uniformly to \( f_0 \) on \( K \). Passing to a subsequence we can assume that

\[
\max_{z \in K} |f_k(z) - f_0(z)| \geq \delta > 0
\]

for \( j = 1, 2, \ldots \). Since the set \( \{f_k : k = 1, 2, \ldots\} \) is a normal family, we can after passage to another subsequence if necessary, assume that \( f_{k_j} \to f \) in \( H(\Omega) \) as \( j \to \infty \) for some \( f \in H(\Omega) \). Since \( \lim_{k \to \infty} f_k(x) = f_0(x) \) for \( x > 0 \), we have that \( f(x) = f_0(x) \) for \( x > 0 \). By uniqueness of analytic continuation we conclude that \( f(z) = f_0(z) \) for \( z \in \Omega \). Thus \( f_{k_j} \to f_0 \) in \( H(\Omega) \) as \( j \to \infty \), which contradicts (5.5).

Recall that \( f_k' \to f_0' \) in \( H(\Omega) \) if \( f_k \to f_0 \) in \( H(\Omega) \). From Theorem 5.2 we thus have that

\[
\lim_{k \to +\infty} k! E^{(j)}_{a,k}(z) = \int_0^\infty (-t)^j e^{-t} g_a^{(j)}(-zt) dt
\]

uniformly on compact subsets of \( \mathbb{C} \setminus (\infty, 0] \) for every \( j \geq 0 \).

The exponential integral is the function \( E_1 \) defined by

\[
E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt, \quad z \in \mathbb{C} \setminus (\infty, 0],
\]

where the path of integration is the ray from \( z \) to \( \infty \) parallel to the real axis in the direction of increasing real part. Parameterizing this latter ray we see that

\[
E_1(z) = \int_0^\infty \frac{e^{-(z+t)}}{z+t} dt = e^{-z} \int_0^\infty \frac{e^{-t}}{z+t} dt
\]

for \( z \in \mathbb{C} \setminus (\infty, 0] \).
Lemma 5.5. The exponential integral admits the estimate
\[ |E_1(z)| \leq E_1(x) \leq \log \left( 1 + \frac{1}{x} \right) e^{-x} \]
for \( z = x + iy \) with \( x > 0 \) and \( y \in \mathbb{R} \).

Proof. Let \( z = x + iy \) with \( x > 0 \) and \( y \in \mathbb{R} \). We proceed to estimate \( E_1(z) \). Recall formula (5.6). Observe that \( |z + t| \geq x + t > 0 \) for \( t > 0 \). By the triangle inequality we have that
\[
|E_1(z)| \leq e^{-x} \int_0^\infty \frac{e^{-t}}{|z + t|} dt \leq e^{-x} \int_0^\infty \frac{e^{-t}}{x + t} dt = E_1(x),
\]
where the last equality again follows by (5.6). This proves the leftmost inequality in the lemma.

We proceed to estimate \( E_1(x) \). Recall formula (5.6). An integration by parts gives that
\[
e^x E_1(x) = \int_0^\infty \frac{e^{-t}}{x + t} dt = -\log(x) + \int_0^\infty \log(x + t)e^{-t} dt.
\]
Observe that \( \int_0^\infty e^{-t} dt = 1 \). From the previous equality we have that
\[
e^x E_1(x) = \int_0^\infty \log(1 + t/x)e^{-t} dt.
\]
An application of the inequality between arithmetic and geometric means (Jensen’s inequality) now gives that
\[
\exp \left( e^x E_1(x) \right) \leq \int_0^\infty \exp(\log(1 + t/x))e^{-t} dt = \int_0^\infty (1 + t/x)e^{-t} dt = 1 + 1/x,
\]
where in the last equality we have used that \( \int_0^\infty te^{-t} dt = 1 \) (see [21, Theorem 3.3]). Thus
\[
e^x E_1(x) \leq \log(1 + 1/x)
\]
since the logarithm function is increasing. This proves the rightmost inequality in the lemma. □

We mention that the rightmost inequality in Lemma 5.5 is a result of Gautschi [11]; see also [1, formula (5.1.20)]. We have merely supplied a simplified proof.

We shall now investigate the limit
(5.7) \[ L_\alpha(z) = \int_0^\infty e^{-t} g_\alpha(-zt) dt, \quad z \in \mathbb{C} \setminus (-\infty, 0], \]
appearing in Theorem 5.2. It is evident from Theorem 5.2 that the function \( L_\alpha \) is analytic in \( \mathbb{C} \setminus (-\infty, 0] \).

We define powers in the usual way using a logarithm in \( \mathbb{C} \setminus (-\infty, 0] \) which is real on the positive real axis.

Theorem 5.3. Let \( \alpha > -1 \) and let \( L_\alpha \) be as in (5.7). Then
\[
L_\alpha(z) = \frac{1}{z^{\alpha+1}} \int_{1/z}^\infty t^{-(\alpha+1)} e^t E_1(t) dt
\]
for \( z \in \mathbb{C} \setminus (-\infty, 0] \), where the path of integration is the ray from \( 1/z \) to \( \infty \) parallel to the real axis in the direction of increasing real part.
Proof. Observe that the integral appearing in the theorem is absolutely convergent by Lemma 5.5. We consider the auxiliary function
\[ h(z) = \frac{1}{z^{\alpha+1}} \int_{1/z}^{\infty} t^{-\alpha-1} e^{t} E_1(t) \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0], \]
where the path of integration is the ray from $1/z$ to $\infty$ parallel to the real axis in the direction of increasing real part. From standard complex analysis we have that the function $h$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

We proceed to show that $L_\alpha(x) = h(x)$ for $x > 0$. Let $x > 0$. By the integral formula in Lemma 3.1 we have that
\[ L_\alpha(x) = \int_0^\infty e^{-t} \left( \int_0^1 s^\alpha \frac{e^{-t}}{1 + xts} \, ds \right) \, dt = \int_0^1 \frac{s^\alpha}{x} \left( \int_0^\infty e^{-t} \frac{1}{1 + xst} \, dt \right) \, ds, \]
where the last equality follows by a change in order of integration. By formula (5.6) we have that
\[ \int_0^\infty e^{-t} \frac{1}{1 + xst} \, dt = \frac{1}{xs} e^{1/(xs)} E_1(1/(xs)) \]
for $s > 0$. Thus
\[ L_\alpha(x) = \frac{1}{x} \int_0^1 s^\alpha e^{1/(xs)} E_1(1/(xs)) \, ds = \frac{1}{x^{\alpha+1}} \int_{1/x}^{\infty} y^{-\alpha-1} e^y E_1(y) \, dy, \]
where the last equality follows by the change of variables $y = 1/xs$. This proves that $L_\alpha(x) = h(x)$ for $x > 0$.

By uniqueness of analytic continuation we conclude that $L_\alpha(z) = h(z)$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$. This completes the proof of the theorem. \qed

We mention that the function $L_\alpha$ satisfies the differential equation
\[ zL_\alpha'(z) + (\alpha + 1)L_\alpha(z) = \int_0^\infty e^{-t} \frac{1}{1 + zt} \, dt, \quad z \in \mathbb{C} \setminus (-\infty, 0], \]
and $L_\alpha(0) = 1/(\alpha + 1)$. In fact, the function $g_\alpha$ satisfies the differential equation
\[ zg_\alpha'(z) + (\alpha + 1)g_\alpha(z) = 1/(1 - z), \quad z \in \mathbb{C} \setminus [1, \infty), \]
which leads to (5.8). We omit the details.

**References**


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