A characteristic operator function for the class of $n$-hypercontractions

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Abstract

We consider a class of bounded linear operators on Hilbert space called $n$-hypercontractions which relates naturally to adjoint shift operators on certain vector-valued standard weighted Bergman spaces on the unit disc. In the context of $n$-hypercontractions in the class $C_0$, we introduce a counterpart to the so-called characteristic operator function for a contraction operator. This generalized characteristic operator function $W_{n,T}$ is an operator-valued analytic function in the unit disc whose values are operators between two Hilbert spaces of defect type. Using an operator-valued function of the form $W_{n,T}$, we parametrize the wandering subspace for a general shift invariant subspace of the corresponding vector-valued standard weighted Bergman space. The operator-valued analytic function $W_{n,T}$ is shown to act as a contractive multiplier from the Hardy space into the associated standard weighted Bergman space.

Keywords: Characteristic operator function; $n$-Hypercontraction; Wandering subspace; Standard weighted Bergman space; Reproducing kernel function

0. Introduction

Let us first describe a class of vector-valued standard weighted Bergman spaces that will play an important role in this paper. Let $n \geq 1$ be an integer and let $\mathcal{E}$ be a general not necessarily

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separable Hilbert space. We denote by $A_n(\mathcal{E})$ the Hilbert space of all $\mathcal{E}$-valued analytic functions

$$f(z) = \sum_{k \geq 0} a_k z^k, \quad z \in \mathbb{D},$$

in the unit disc $\mathbb{D}$ with finite norm

$$\|f\|_{A_n}^2 = \sum_{k \geq 0} \|a_k\|^2 \mu_{n;k},$$

where $\mu_{n;k} = 1/(k+n-1)$ for $k \geq 0$. Here the Taylor coefficients $a_k$ in (0.1) are elements in $\mathcal{E}$.

The weight sequence $\{\mu_{n;k}\}_{k \geq 0}$ is naturally identified as a sequence of moments of a certain radial measure $d\mu_n$ on the closed unit disc in the sense that

$$\mu_{n;k} = \int_{\overline{\mathbb{D}}} |z|^{2k} d\mu_n(z) = 1\left(\frac{k+n-1}{k}\right), \quad k \geq 0.$$  

For $n \geq 2$ the measure $d\mu_n$ is given by

$$d\mu_n(z) = (n-1)(1-|z|^2)^{n-2} dA(z), \quad z \in \mathbb{D},$$

where $d\mu_2(z) = dA(z) = dx\,dy/\pi$, $z = x + iy$, is the usual planar Lebesgue area measure normalized so that the unit disc $\mathbb{D}$ is of unit area. The measure $d\mu_1$ is the normalized Lebesgue arc length measure on the unit circle $\mathbb{T} = \partial \mathbb{D}$. The norm of $A_n(\mathcal{E})$ can also be expressed as

$$\|f\|_{A_n}^2 = \lim_{r \to 1} \frac{1}{\mathbb{D}} \int_{\overline{\mathbb{D}}} \|f(rz)\|^2 d\mu_n(z), \quad f \in A_n(\mathcal{E}).$$

The shift operator $S_n$ on the space $A_n(\mathcal{E})$ is defined by

$$(S_n f)(z) = zf(z) = \sum_{k \geq 1} a_{k-1} z^k, \quad z \in \mathbb{D},$$

for $f \in A_n(\mathcal{E})$ given by (0.1). It is easy to see that the shift operator $S_n$ is bounded on $A_n(\mathcal{E})$ of norm equal to 1 (the weight sequence $\{\mu_{n;k}\}_{k \geq 0}$ is decreasing and the ratio $\mu_{n;k+1}/\mu_{n;k}$ tends to 1 as $k \to \infty$). The adjoint operator $S_n^*$ of $S_n$ has the form

$$(S_n^* f)(z) = \sum_{k \geq 0} \frac{\mu_{n;k+1}}{\mu_{n;k}} a_{k+1} z^k, \quad z \in \mathbb{D},$$

where the function $f \in A_n(\mathcal{E})$ is given by (0.1).

Let $\mathcal{I}$ be a shift invariant subspace of $A_n(\mathcal{E})$. By this we mean that $\mathcal{I}$ is a closed subspace of $A_n(\mathcal{E})$ which is invariant under the shift operator $S_n$ in the sense that $S_n(\mathcal{I}) \subset \mathcal{I}$. The wandering subspace $\mathcal{E}_\mathcal{I}$ for $\mathcal{I}$ is the subspace

$$\mathcal{E}_\mathcal{I} = \mathcal{I} \ominus S_n(\mathcal{I})$$
of \( \mathcal{I} \). The subspace \( \mathcal{E}_I \) has the property that \( \mathcal{E}_I \perp S_n^k(\mathcal{E}_I) \) for \( k \geq 1 \), which is often used as the defining property for a wandering subspace. The notion of a wandering subspace is often accredited to Halmos [12] and was used as an important concept in his description of the shift invariant subspaces of the Hardy space \( A_1(\mathcal{E}) \) using operator-valued inner functions.

In the genuine Bergman space case \( n \geq 2 \) it is known that the wandering subspace \( \mathcal{E}_I \) for a shift invariant subspace \( \mathcal{I} \) of \( A_n(\mathcal{E}) \) can have dimension equal to any positive integer or \( +\infty \) even in the case \( \mathcal{E} = \mathbb{C} \) of scalar-valued functions. This was first proved by Apostol et al. [5] using dual algebras, and later more explicit constructions have been found by Hedenmalm et al. [19] and others.

In later developments the notion of a wandering subspace has proved to be a useful concept to study shift invariant subspaces in a Bergman space context. In the scalar case of invariant subspaces generated by zero sets Hedenmalm [13–15] has shown that functions in the wandering subspace also called Bergman inner functions can be used to divide out zeroes of functions in the subspace for \( n = 2, 3 \). For \( n = 1 \) we are in the Hardy space context, and for \( n \geq 4 \) such a theorem fails (see [18]).

A related question which has attracted much attention is to what extent the wandering subspace \( \mathcal{E}_I \) generates the whole invariant subspace \( \mathcal{I} \) in the sense that

\[
\mathcal{I} = [\mathcal{E}_I] = \bigvee_{k \geq 0} S_n^k(\mathcal{E}_I);
\]

we use \([\mathcal{F}]\) to denote the smallest (closed) shift invariant subspace containing the set \( \mathcal{F} \). In our context of the Bergman spaces \( A_n(\mathcal{E}) \) the approximation relation (0.4) is known to hold true for a general shift invariant subspace \( \mathcal{I} \) of \( A_n(\mathcal{E}) \) for the values \( n = 1, 2, 3 \), and is most probably false in general for \( n \geq 4 \) (see [17,18]). The case \( n = 1 \) here is the Hardy space case mentioned earlier where a parametrization of the shift invariant subspaces is available. In the case of an unweighted Bergman space \( n = 2 \) the approximation relation (0.4) for a general shift invariant subspace was first established by Aleman et al. [3] using function theoretic properties of the biharmonic Green function for the unit disc; see also [16, Section 3.6] and [20]. Later Shimorin [24] found a more general result which applies to a more general class of pure operators satisfying a certain operator inequality satisfied by the Bergman shift operator \( S_2 \). The case \( n = 3 \) is due to Shimorin [25]. In the case \( n = 2 \) some summability results stronger than (0.4) are known to hold true (see [21]).

Despite all the developments indicated above there are few explicit examples known of Bergman inner functions or, what is the same, wandering subspaces in the Bergman spaces. It is the purpose of the present paper to give a parametrization of the wandering subspace for a general shift invariant subspace in the context of the vector-valued standard weighted Bergman spaces \( A_n(\mathcal{E}) \) described above. This parametrization is done in terms of certain operator theoretic quantities known as defect spaces, and some explicit formulas are obtained in the process. Let us now proceed to describe the content of the present paper.

By a Hilbert space we mean a general not necessarily separable complex Hilbert space. We denote by \( \mathcal{L}(\mathcal{H}) \) the space of all bounded linear operators on a Hilbert space \( \mathcal{H} \). Let \( n \geq 1 \) be an integer. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is called an \( n \)-hypercontraction if the operator inequality

\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^* T^k \geq 0 \quad \text{in} \ \mathcal{L}(\mathcal{H})
\]
holds for every $1 \leq m \leq n$. In this terminology a 1-hypercontraction is a contraction, and for $n \geq 2$ the class of $n$-hypercontractions defines a more restricted class of operators.

The class of $n$-hypercontractions was first introduced by Agler [1,2]. A principal result of [2] concerns the description of a general $n$-hypercontraction. An operator $T \in \mathcal{L}(\mathcal{H})$ is an $n$-hypercontraction if and only if it is unitarily equivalent to the restriction to an invariant subspace of an operator of the form $S_n^* \oplus U$, where $S_n$ is the shift operator on a Bergman space $A_n(\mathcal{E})$ and $U$ is an isometry. A special case of this result is the well-known fact that an operator $T \in \mathcal{L}(\mathcal{H})$ is a contraction if and only if it is part of an operator of the form $S_1^* \oplus U$, where $S_1$ is the shift operator on a Hardy space $A_1(\mathcal{E})$ and $U$ is an isometry, which is often accredited to Rota, de Branges, Rovnyak, Sz.-Nagy and Foias (see [26, Section I.10.1]).

Let us recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to the class $C_0$ if $\lim_{k \to \infty} T^k = 0$ in the strong operator topology meaning that $\lim_{k \to \infty} T^k x = 0$ in $\mathcal{H}$ for every $x \in \mathcal{H}$ (see [26, Section II.4]). For operators from the class $C_0$, the isometric term $U$ in the description in the previous paragraph vanishes and one has that an operator $T \in \mathcal{L}(\mathcal{H})$ is an $n$-hypercontraction such that $\lim_{k \to \infty} T^k = 0$ in the strong operator topology if and only if it is a restriction of the adjoint shift operator $S_n^*$ to an invariant subspace.

We shall need some more notations related to an $n$-hypercontraction $T \in \mathcal{L}(\mathcal{H})$. We consider the defect operators

$$D_{m,T} = \left( \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^* T^k \right)^{1/2} \quad \text{in } \mathcal{L}(\mathcal{H})$$

for $1 \leq m \leq n$, where the positive square root is used. The defect space $D_{m,T}$ is defined as the closure in $\mathcal{H}$ of the range of the operator $D_{m,T}$, that is, $D_{m,T} = \overline{\text{range } D_{m,T}}$. For $n = 1$ and $T \in \mathcal{L}(\mathcal{H})$ a contraction operator we write also

$$D_T = D_{1,T} = (I - T^* T)^{1/2} \quad \text{in } \mathcal{L}(\mathcal{H})$$

and $\mathcal{D}_T = \overline{\text{range } D_{T}}$ for the defect operator and the associated defect space.

In recent work [22] we have revisited the operator model theory for the class of $n$-hypercontractions. It turns out that there is a canonical way to model an $n$-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ as part of an operator of the form $S_n^* \oplus U$, where $U$ is an isometry. For $x \in \mathcal{H}$ we consider the $\mathcal{D}_{n,T}$-valued analytic function $V_{n,x}$ defined by the formula

$$(V_{n,x})(z) = D_{n,T}(I - z T)^{-n} x = \sum_{k \geq 0} \binom{k+n-1}{k} (D_{n,T} T^k x) z^k, \quad z \in \mathbb{D}. \quad (0.5)$$

It turns out that if $T \in \mathcal{L}(\mathcal{H})$ is an $n$-hypercontraction such that $\lim_{k \to \infty} T^k = 0$ in the strong operator topology, then the map $V_n : x \mapsto V_{n,x}$ given by (0.5) is an isometry

$$V_n : \mathcal{H} \to A_n(\mathcal{D}_{n,T})$$

of $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T})$ satisfying the intertwining relation

$$V_n T = S_n^* V_n.$$
In this way an $n$-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ in the class $C_0$ is naturally modeled as part of the adjoint shift operator $S_n^*$ on the Bergman space $A_n(D_{n,T})$. Full details of this construction can be found in [22, Sections 6 and 7].

We mention that construction of operator models of this type is a topic of current interest in multi-variable operator theory with recent contributions by Ambrozie et al. [4] and Arazy and Engliš [6]. Operator models of this type also form an integral part in recent work on constrained von Neumann inequalities by Badea and Cassier [8].

In this paper we shall consider in some more detail the subspace $I_{n,T} = A_n(D_{n,T}) \ominus V_n(\mathcal{H})$ of $A_n(D_{n,T})$. Since the range $V_n(\mathcal{H})$ is invariant for $S_n^*$, its orthogonal complement $I_{n,T}$ is invariant for the shift operator $S_n$. In other words, the space $I_{n,T}$ is a shift invariant subspace of $A_n(D_{n,T})$. The wandering subspace $E_{n,T}$ for $I_{n,T}$ is the subspace $E_{n,T} = I_{n,T} \ominus S_n(I_{n,T})$ of $I_{n,T}$. To present our parametrization of the wandering subspace $E_{n,T}$ for $I_{n,T}$ we need some more notations.

Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction. We denote by $\mathcal{H}_n$ the space $\mathcal{H}$ equipped with the equivalent norm

$$
\|x\|_n^2 = \sum_{k=0}^{n-1} (-1)^k \left( \binom{n}{k+1} \right) \|T^k x\|^2 = \|x\|^2 + \sum_{k=1}^{n-1} \|D_k T x\|^2, \quad x \in \mathcal{H}
$$

(see Lemma 3.1). It turns out that the operator

$$
TT^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{*k} T^k \right) \quad \text{in} \quad \mathcal{L}(\mathcal{H})
$$

is self-adjoint in $\mathcal{L}(\mathcal{H}_n)$ and has its spectrum contained in the closed unit interval $[0, 1]$ (see Lemma 3.3). We denote by $Q_{n,T}$ the operator

$$
Q_{n,T} = \left( I - TT^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{*k} T^k \right) \right)^{1/2} \quad \text{in} \quad \mathcal{L}(\mathcal{H}),
$$

where the positive square root is computed in $\mathcal{L}(\mathcal{H}_n)$. By $D_{n,T}^*$ we denote the closure in $\mathcal{H}$ of the range of this operator $Q_{n,T}$, and we equip this space $D_{n,T}^*$ with the norm $\| \cdot \|_n$ defined by (0.6). It turns out that we have the equality

$$
T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{*k} T^k \right) Q_{n,T} = D_{n,T} T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{*k} T^k \right)
$$

(0.7) in $\mathcal{L}(\mathcal{H})$ (see Lemma 3.4).
In the case \( n = 1 \) of a contraction operator \( T \in \mathcal{L}(\mathcal{H}) \) the notions in the previous paragraph specialize to well-known objects. The norm \( \| \cdot \|_1 \) defined by (0.6) coincides with the usual norm of \( \mathcal{H} \). The operator \( Q_{1,T} \) is the defect operator for the adjoint operator \( T^* \), that is, \( Q_{1,T} = D_{T^*} \), and the space \( D_{1,T}^* \) is the defect space \( D_{1,T}^* = D_{T^*} \) for \( T^* \). The equality (0.7) reduces to the well-known formula \( T^* D_{T^*} = D_{T^*} T^* \) for defect operators.

For an \( n \)-hypercontraction \( T \in \mathcal{L}(\mathcal{H}) \) we consider the operator-valued analytic function \( W_{n,T} \) in the unit disc \( \mathbb{D} \) defined by the formula

\[
W_{n,T}(z) = \left[ -T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{sk} T^k \right) + z D_{n,T} \left( \sum_{k=1}^{n} (I - zT)^{-k} \right) Q_{n,T} \right]_{D_{n,T}^*},
\]

\( z \in \mathbb{D} \).

Notice that by (0.7) the values \( W_{n,T}(z) \) attained by this function \( W_{n,T} \) are operators in \( \mathcal{L}(D_{n,T}^*, D_{n,T}) \), that is, bounded linear operators from \( D_{n,T}^* \) into \( D_{n,T} \).

We remark that in the case \( n = 1 \) of a contraction operator \( T \in \mathcal{L}(\mathcal{H}) \) we get the so-called characteristic operator function

\[
W_T(z) = W_{1,T}(z) = \left[ -T^* + z D_T (I - zT)^{-1} D_{T^*} \right]_{D_{T^*}}, \quad z \in \mathbb{D},
\]

whose values are operators in \( \mathcal{L}(D_{T^*}, D_T) \) which has been studied by Sz.-Nagy and Foias (see [26, Chapter VI]).

We have the following description of the wandering subspace \( E_{n,T} \). A function \( f \) in \( A_n(D_{n,T}) \) belongs to the wandering subspace \( E_{n,T} \) for \( I_{n,T} \) if and only if it has the form

\[
f(z) = W_{n,T}(z)x, \quad z \in \mathbb{D}, \tag{0.8}
\]

for some element \( x \in D_{n,T}^* \). Furthermore, we have the norm equality

\[
\| f \|_{A_n}^2 = \| x \|_{n,x}^2, \quad x \in D_{n,T}^*,
\]

when \( f \) is given by (0.8). We recall that the norm \( \| \cdot \|_n \) is defined by (0.6). This parametrization of the wandering subspace \( E_{n,T} \) for \( I_{n,T} \) is the result of Theorem 3.3 in this paper. The proof of Theorem 3.3 proceeds in several steps and takes up Sections 2 and 3 in the paper.

It turns out that the operator-valued analytic function \( W_{n,T} \) has a certain multiplier property in that it acts as a contractive multiplier from the Hardy space \( A_1(D_{n,T}^*) \) into the Bergman space \( A_n(D_{n,T}) \) (see Theorem 4.1). This contractive multiplier property leads in turn to an estimate

\[
W_{n,T}(z)W_{n,T}(z)^* \leq \frac{1}{(1 - |z|^2)^{n-1}} I_{D_{n,T}} \quad \text{in } \mathcal{L}(D_{n,T}), \quad z \in \mathbb{D} \tag{0.9}
\]

(see Theorem 4.2).

In the case \( n = 1 \) of a contraction operator \( T \in \mathcal{L}(\mathcal{H}) \) in the class \( C_0 \), it is known that the characteristic operator function \( W_T \) is an isometric multiplier from the Hardy space \( A_1(D_{T^*}) \) into the Hardy space \( A_1(D_T) \) with range equal to \( I_{1,T} \) (see Corollary 4.1). Here the inequality (0.9) says that the characteristic operator function \( W_T \) attains contractive values (see Corollary 4.2).
We mention also that the contractive multiplier property of $W_{n,T}$ and the inequality (0.9) generalize to a vector-valued context known properties of Bergman inner functions going back to Hedenmalm [13–15] for $n = 2, 3$.

As we have indicated above the previous considerations apply also to general shift invariant subspaces in the Bergman spaces $A_n(\mathcal{E})$. Let $\mathcal{I}$ be a shift invariant subspace of $A_n(\mathcal{E})$. Now the orthogonal complement

$$\mathcal{H} = A_n(\mathcal{E}) \ominus \mathcal{I}$$

of $\mathcal{I}$ is invariant for $S_n^*$ and we set $T = S_n^*|_\mathcal{H}$. The operator $T \in L(\mathcal{H})$ is an $n$-hypercontraction in the class $C_0$. As above we can model this operator $T$ by means of the map $V_n$ given by (0.5). Furthermore, by a uniqueness property of this representation, there exists an isometry $\hat{V}_n : D_{n,T} \to \mathcal{E}$ such that every function $f \in \mathcal{H}$ admits the representation

$$f(z) = \hat{V}_n D_{n,T} (I - zT)^{-n} f, \quad z \in \mathbb{D}. \quad (0.10)$$

The isometry $\hat{V}_n : D_{n,T} \to \mathcal{E}$ is uniquely determined by (0.10) and is given by

$$\hat{V}_n : D_{n,T} f \mapsto f(0) \quad \text{for } f \in \mathcal{H}.$$

Full details of this construction can be found in [22, Sections 6 and 7].

We write $\tilde{\mathcal{E}} = \hat{V}_n(D_{n,T}) \subset \mathcal{E}$. The map $\hat{V}_n$ naturally extends to an isometry

$$\hat{V}_n : A_n(D_{n,T}) \to A_n(\mathcal{E})$$

of $A_n(D_{n,T})$ into $A_n(\mathcal{E})$ with range equal to $A_n(\tilde{\mathcal{E}})$ by setting

$$(\hat{V}_n f)(z) = \hat{V}_n(f(z)), \quad z \in \mathbb{D},$$

for $f \in A_n(D_{n,T})$.

The shift invariant subspace $\mathcal{I}$ now decomposes as an orthogonal sum

$$\mathcal{I} = A_n(\mathcal{E} \ominus \tilde{\mathcal{E}}) \oplus \hat{V}_n(I_{n,T})$$

(see Theorem 5.1), and we can identify the wandering subspace $\mathcal{E}_\mathcal{I}$ for $\mathcal{I}$ as the orthogonal sum

$$\mathcal{E}_\mathcal{I} = (\mathcal{E} \ominus \tilde{\mathcal{E}}) \oplus \hat{V}_n(I_{n,T}).$$

By our previous description of the wandering subspace $\mathcal{E}_{n,T}$ for $I_{n,T}$ we have that a function $f$ in $A_n(\mathcal{E})$ belongs to the wandering subspace $\mathcal{E}_\mathcal{I}$ for $\mathcal{I}$ if and only if it has the form

$$f(z) = a_0 + \hat{V}_n W_{n,T}(z) g, \quad z \in \mathbb{D}, \quad (0.11)$$

for some elements $a_0 \in \mathcal{E} \ominus \tilde{\mathcal{E}}$ and $g \in D_{n,T}^*$. Furthermore, we have the norm equality $\|f\|_{A_n}^2 = \|a_0\|^2 + \|g\|^2_n$ for $f \in A_n(\mathcal{E})$ of the form (0.11) (see Theorem 5.2).

As a byproduct of our considerations we obtain in an explicit form a parametrization of the shift invariant subspaces of the Hardy space $A_1(\mathcal{E})$ in case of a general not necessarily separable
Hilbert space \( E \) (see Corollaries 4.1 and 5.1). We discuss also some relations between the index of a shift invariant subspace and the defect indexes of the adjoint shift restricted to its orthogonal complement (see Corollary 5.2 and Proposition 5.1).

We wish to mention that a source of inspiration for the work presented in this article has been the survey paper [9] by Ball and Cohen.

1. Preliminaries

Let us first recall some constructions developed in the context of so-called wandering subspace theorems in the papers [23,24]. The reason to include this discussion here is that it provides a motivation for some of the arguments we shall use in later sections.

Let \( T \in \mathcal{L}(\mathcal{H}) \) be an injective operator. It is easy to see that then the following statements are equivalent:

- The operator \( T \) has closed range \( T(\mathcal{H}) \).
- The operator \( T \) is bounded from below in the sense that \( \|Tx\|^2 \geq c\|x\|^2 \) for \( x \in \mathcal{H} \) and some positive constant \( c \).
- The operator \( T \) is left-invertible.

Let now \( T \in \mathcal{L}(\mathcal{H}) \) be an operator satisfying any of these conditions. The wandering subspace for the operator \( T \in \mathcal{L}(\mathcal{H}) \) is the subspace

\[
E = \mathcal{H} \ominus T(\mathcal{H}) = \ker T^*
\]

of \( \mathcal{H} \). The operator \( L = (T^*T)^{-1} T^* \) in \( \mathcal{L}(\mathcal{H}) \) is the left-inverse of \( T \) with kernel \( E \):

\[
LT = I \quad \text{in} \quad \mathcal{L}(\mathcal{H}) \quad \text{and} \quad \ker L = \ker T^* = \mathcal{E}.
\]

The operator

\[
P = I - TL \quad \text{in} \quad \mathcal{L}(\mathcal{H})
\]

is the orthogonal projection of \( \mathcal{H} \) onto \( E \). Indeed, the operator \( TL = T(T^*T)^{-1} T^* \) is self-adjoint, idempotent and has range equal to \( T(\mathcal{H}) \).

We shall also have use of the operator

\[
T' = L^* = T(T^*T)^{-1} \quad \text{in} \quad \mathcal{L}(\mathcal{H}).
\]

The operator \( T' \) turns out to have some properties dual to those of \( T \) (see [21,24]). We notice that

\[
(T')^*T' = (T^*T)^{-1} T^* T(T^*T)^{-1} = (T^*T)^{-1} \quad \text{in} \quad \mathcal{L}(\mathcal{H}). \tag{1.1}
\]

Let us now specialize to the shift operator \( S_n \) on the Bergman space \( A_n(\mathcal{E}) \). The formulas (0.2) and (0.3) make evident that the operator \( S_n^* S_n \) on \( A_n(\mathcal{E}) \) acts as

\[
(S_n^* S_n)(f)(z) = \sum_{k \geq 0} \frac{\mu_n; k+1}{\mu_n; k} a_k z^k, \quad z \in \mathbb{D},
\]
where $f \in A_n(\mathcal{E})$ is given by (0.1). We notice that
\[
\|S_nf\|_{A_n}^2 = \langle S_n^* S_n f, f \rangle_{A_n} = \sum_{k \geq 0} \|a_k\|^2 \mu_{n,k+1},
\]
where $f \in A_n(\mathcal{E})$ is given by (0.1). It is straightforward to verify that the quotient $\mu_{n,k+1}/\mu_{n,k}$ is increasing in $k \geq 0$. As a result we have that the operator $S_n$ is bounded from below with constant $c = 1/n$.

A computation shows that the operator $L_n = (S_n^* S_n)^{-1} S_n^*$ on $A_n(\mathcal{E})$ acts as
\[
(L_n f)(z) = \frac{f(z) - f(0)}{z} = \sum_{k \geq 0} a_{k+1} z^k, \quad z \in \mathbb{D},
\]
where $f \in A_n(\mathcal{E})$ is given by (0.1). The operator $S'_n = L_n^* = S_n (S_n^* S_n)^{-1}$ is the weighted shift operator on $A_n(\mathcal{E})$ acting as
\[
(S'_n f)(z) = \sum_{k \geq 1} \frac{\mu_{n,k-1}}{\mu_{n,k}} a_{k-1} z^k, \quad z \in \mathbb{D}, \tag{1.2}
\]
where $f \in A_n(\mathcal{E})$ is given by (0.1). We notice that
\[
\|S'_n f\|_{A_n}^2 = \sum_{k \geq 1} \mu_{n,k-1}^2 \|a_{k-1}\|^2 = \sum_{k \geq 0} \mu_{n,k+1}^2 \|a_k\|^2, \tag{1.3}
\]
where $f \in A_n(\mathcal{E})$ is given by (0.1).

Sums involving binomial coefficients will appear in our calculations. For the sake of easy reference we record the following lemma.

**Lemma 1.1.** Let $\mu_{n,k} = 1/(\binom{k+n-1}{k})$ for $n \geq 1$ and $k \geq 0$. Then
\[
\min(n-1,k) \sum_{j=0}^{\min(n-1,k)} (-1)^j \binom{n}{j+1} \frac{1}{\mu_{n,k-j}} = \frac{1}{\mu_{n,k+1}}.
\]

**Proof.** A computation shows that
\[
\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} z^k = \frac{1 - (1 - z)^n}{z}.
\]
The sum in the lemma equals the $k$th Taylor coefficient of the function
\[
\frac{1 - (1 - z)^n}{z} = \frac{1}{z} \left( \frac{1}{(1 - z)^n} - 1 \right) = \sum_{k \geq 0} \frac{1}{\mu_{n,k+1}} z^k, \quad z \in \mathbb{D}.
\]
This yields the conclusion of the lemma. \qed
We shall need the following norm equality.

**Proposition 1.1.** Let \( f \in A_n(\mathcal{E}) \) be given by (0.1). Then

\[
\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \| S_n^{*k} f \|^2_{A_n} = \sum_{k \geq 0} \frac{\mu_{n;k}^2}{\mu_{n;k+1}} \| a_k \|^2.
\]

**Proof.** By (0.3) we have that

\[
\| S_n^{*j} f \|^2_{A_n} = \sum_{k \geq 0} \frac{\mu_{n;k+j}^2}{\mu_{n;k}} \| a_{k+j} \|^2
\]

for \( j \geq 0 \). Summing these equalities we have by a change of order of summation that

\[
\sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \| S_n^{*j} f \|^2_{A_n} = \sum_{k \geq 0} \left( \sum_{j=0}^{\min(n-1,k)} (-1)^j \binom{n}{j+1} \frac{1}{\mu_{n;k-j}} \right) \| a_k \|^2 \mu_{n;k}
\]

\[= \sum_{k \geq 0} \frac{\mu_{n;k}^2}{\mu_{n;k+1}} \| a_k \|^2,
\]

where the last equality follows by Lemma 1.1. \( \square \)

We remark that the sums in Proposition 1.1 equal \( \| S_n' f \|^2_{A_n} \) by (1.3). By a polarization argument we conclude that

\[(S_n^* S_n)^{-1} = (S_n')^* S_n' = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} S_n^k S_n^{*k} \in \mathcal{L}(A_n(\mathcal{E})), \quad (1.4)
\]

where the first equality follows by (1.1).

### 2. A first description of the wandering subspace

We use the same basic notations as in the introduction. The operator \( V_n \) in \( \mathcal{L}(\mathcal{H}, A_n(D_n,T)) \) is defined by (0.5),

\[\mathcal{I}_{n,T} = A_n(D_{n,T}) \oplus V_n(\mathcal{H}) \quad \text{and} \quad \mathcal{E}_{n,T} = \mathcal{I}_{n,T} \ominus S_n(\mathcal{I}_{n,T}).\]

In this section we shall give a first description of the wandering subspace \( \mathcal{E}_{n,T} \) for \( \mathcal{I}_{n,T} \) in Theorem 2.1. First we need a few lemmas.

**Lemma 2.1.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be an \( n \)-hypercontraction in the class \( C_0 \). Then the operator

\[P_n = I - V_n V_n^* \quad \text{in} \ \mathcal{L}(A_n(D_{n,T}))
\]

is the orthogonal projection of \( A_n(D_{n,T}) \) onto \( \mathcal{I}_{n,T} \).
Proof. We consider the operator \( Q_n = V_n V_n^* \). Recall that \( V_n : \mathcal{H} \to A_n(\mathcal{D}_n, T) \) is an isometry (see [22, Section 7]). It is straightforward to see that the operator \( Q_n \) is self-adjoint, idempotent and has range equal to \( V_n(\mathcal{H}) \). Accordingly the operator \( Q_n \) is the orthogonal projection of \( A_n(\mathcal{D}_n, T) \) onto \( V_n(\mathcal{H}) \), and \( P_n = I - Q_n \) is the orthogonal projection of \( A_n(\mathcal{D}_n, T) \) onto \( \mathcal{I}_{n,T} = A_n(\mathcal{D}_n, T) \ominus V_n(\mathcal{H}) \). □

We next compute the operator \( V_n^* \).

Lemma 2.2. Let \( T \in \mathcal{L}(\mathcal{H}) \) be an \( n \)-hypercontraction in the class \( C_0 \). Then the adjoint operator \( V_n^* : A_n(\mathcal{D}_n, T) \to \mathcal{H} \) acts as

\[
V_n^* f = \sum_{k \geq 0} T^{*k} D_{n,T} a_k \quad \text{weakly in } \mathcal{H},
\]

(2.1)

where \( f \in A_n(\mathcal{D}_n, T) \) is given by (0.1).

Proof. For \( x \in \mathcal{H} \) we have that

\[
\langle V_n^* f, x \rangle = \langle f, V_n x \rangle_{A_n} = \sum_{k \geq 0} \left( a_k, \frac{1}{\mu_{n,k}} \left( D_{n,T} T^k x \right) \right)_{\mu_{n,k}}
\]

\[
= \sum_{k \geq 0} \langle a_k, D_{n,T} T^k x \rangle = \lim_{N \to \infty} \left( \sum_{k=0}^{N} T^{*k} D_{n,T} a_k, x \right).
\]

This gives the conclusion of the lemma. □

We remark that the sum in (2.1) converges in the weak topology in \( \mathcal{H} \).

We shall next compute the operator \( V_n^* (S_n^* S_n)^{-1} V_n = V_n^* (S_n')^* S_n' V_n \).

Lemma 2.3. Let \( T \in \mathcal{L}(\mathcal{H}) \) be an \( n \)-hypercontraction in the class \( C_0 \). Then

\[
V_n^* (S_n^* S_n)^{-1} V_n = V_n^* (S_n')^* S_n' V_n = \sum_{k=0}^{n-1} (-1)^k \left( \binom{n}{k+1} T^{*k} \right) V_n \quad \text{in } \mathcal{L}(\mathcal{H}).
\]

Proof. Recall that the operator \( V_n \) in \( \mathcal{L}(\mathcal{H}, A_n(\mathcal{D}_n, T)) \) is an isometry such that \( V_n T = S_n^* V_n \) (see [22, Sections 6 and 7]). By formula (1.4) we have that

\[
V_n^* (S_n^* S_n)^{-1} V_n = V_n^* (S_n')^* S_n' V_n = V_n^* \left( \sum_{k=0}^{n-1} (-1)^k \left( \binom{n}{k+1} S_n^k S_n^{*k} \right) \right) V_n \quad \text{in } \mathcal{L}(\mathcal{H}).
\]

The intertwining relation \( V_n T = S_n^* V_n \) gives that \( V_n T^k = S_n^{*k} V_n \) for \( k \geq 0 \), and taking adjoints we see that also \( T^{*k} V_n^* = V_n^* S_n^k \) for \( k \geq 0 \). Using these intertwining formulas we now have that...
\[ V_n^*(S_n^*S_n)^{-1}V_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{*k}V_n^*V_nT^k \]
\[ = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{*k}T^k \quad \text{in } \mathcal{L}(\mathcal{H}), \]

where the last equality follows by \( V_n^*V_n = I \) in \( \mathcal{L}(\mathcal{H}) \). This completes the proof of the lemma. \( \square \)

We can now give a first description of the wandering subspace \( \mathcal{E}_{n,T} \).

**Theorem 2.1.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be an \( n \)-hypercontraction in the class \( C_0 \). Then a function \( f \) in \( A_n(D_{n,T}) \) belongs to the wandering subspace \( \mathcal{E}_{n,T} \) for \( \mathcal{I}_{n,T} \) if and only if it has the form

\[ f = a_0 + S_n'V_nx = a_0 + S_n(S_n^*S_n)^{-1}V_nx \]

for some elements \( a_0 \in D_{n,T} \) and \( x \in \mathcal{H} \) such that

\[ D_{n,T}a_0 + T^*\left( \sum_{k=0}^{n-1} (-1)^k \left( \binom{n}{k+1} T^{*k}T^k \right) \right)x = 0. \]  \hspace{1cm} (2.2)

**Proof.** Notice first that

\[ \mathcal{I}_{n,T} = A_n(D_{n,T}) \ominus V_n(\mathcal{H}) = \ker V_n^*, \]

and similarly that

\[ \mathcal{E}_{n,T} = \mathcal{I}_{n,T} \ominus S_n(\mathcal{I}_{n,T}) = \ker (S_n|_{\mathcal{I}_{n,T}})^*. \]

Here

\[ (S_n|_{\mathcal{I}_{n,T}})^* = P_nS_n^* = (I - V_nV_n^*)S_n^* \]

by Lemma 2.1. An element \( f \) in \( A_n(D_{n,T}) \) thus belongs to the wandering subspace \( \mathcal{E}_{n,T} \) for \( \mathcal{I}_{n,T} \) if and only if \( V_n^*f = 0 \) and \( (I - V_nV_n^*)S_n^*f = 0 \).

We consider first the equation \( (I - V_nV_n^*)S_n^*f = 0 \). This equation can be rewritten as \( S_n^*f = V_nV_n^*S_n^*f \). We apply the operator \( (S_n^*S_n)^{-1} \) to obtain that

\[ L_nf = (S_n^*S_n)^{-1}S_n^*f = (S_n^*S_n)^{-1}V_nV_n^*S_n^*f = (S_n^*S_n)^{-1}V_nx, \]

where \( x = V_n^*S_n^*f \in \mathcal{H} \). We now have that

\[ f = a_0 + S_nL_nf = a_0 + S_n(S_n^*S_n)^{-1}V_nx = a_0 + S_n'V_nx, \]  \hspace{1cm} (2.3)

where \( a_0 \in D_{n,T} \) and \( x \in \mathcal{H} \). Conversely, if \( f \) in \( A_n(D_{n,T}) \) is of the form (2.3), then

\[ (I - V_nV_n^*)S_n^*f = (I - V_nV_n^*)V_nx = 0, \]
since $V_n$ is an isometry. We have thus shown that a function $f$ in $A_n(D_{n,T})$ satisfies the equation $(I - V_n V_n^*) S_n^* f = 0$ if and only if it has the form (2.3) for some elements $a_0 \in D_{n,T}$ and $x \in H$.

We shall now compute $V_n^* f$ when $f \in A_n(D_{n,T})$ is of the form (2.3) for some elements $a_0 \in D_{n,T}$ and $x \in H$. Notice that the intertwining relation $V_n T = S_n^* V_n$ gives that $V_n^* S_n = T^* V_n^*$. By Lemma 2.2 we now have that

$$V_n^* f = D_{n,T} a_0 + V_n^* S_n (S_n^* S_n)^{-1} V_n x = D_{n,T} a_0 + T^* V_n^* (S_n^* S_n)^{-1} V_n x.$$  

Recall that the operator $V_n^* (S_n^* S_n)^{-1} V_n$ was computed in Lemma 2.3. We conclude that

$$V_n^* f = D_{n,T} a_0 + T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^k x \right),$$

where $f \in A_n(D_{n,T})$, $a_0 \in D_{n,T}$ and $x \in H$ are related as in (2.3). This gives the conclusion of the theorem. \qed

We shall next compute the norm of a function of the form $f = a_0 + S_n^* V_n x$.

**Theorem 2.2.** Let $T \in \mathcal{L}(H)$ be an $n$-hypercontraction in the class $C_0$. Let $f$ in $A_n(D_{n,T})$ be of the form

$$f = a_0 + S_n^* V_n x$$

for some elements $a_0 \in D_{n,T}$ and $x \in H$. Then

$$\|f\|_{A_n}^2 = \|a_0\|^2 + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \|T^k x\|^2.$$ 

**Proof.** By Lemma 2.3 we have that

$$\|f\|_{A_n}^2 = \|a_0\|^2 + \|S_n^* V_n x\|_{A_n}^2 = \|a_0\|^2 + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \|T^k x\|^2.$$ 

This completes the proof of the theorem. \qed

3. **Parametrization of the wandering subspace $E_{n,T}$**

In this section we shall solve Eq. (2.2) and give a more refined description of the wandering subspace $E_{n,T}$ for $I_{n,T}$ using the operator-valued analytic function $W_{n,T}$.

We shall need some constructions involving the use of defect operators of contractions between Hilbert spaces. Let $A \in \mathcal{L}(H, K)$ be a contraction operator mapping a Hilbert space $H$ into a Hilbert space $K$. Associated to this operator $A$ we have the defect operator $D_A$ defined by

$$D_A = (I - A^* A)^{1/2} \text{ in } \mathcal{L}(H),$$
where the positive square root is used. Notice that
\[ \|x\|^2 = \|Ax\|^2 + \|DAx\|^2, \quad x \in H, \] (3.1)
and that this equality (3.1) can be restated saying that \( I = A^*A + D_A^2 \) in \( \mathcal{L}(H) \). The defect space \( D_A \) for \( A \) is defined as the closure in \( H \) of the range of the operator \( D_A \), that is, \( D_A = \overline{D_A(H)} \). In the same way the adjoint operator \( A^* \in \mathcal{L}(K,H) \) has an associated defect operator
\[ D_{A^*} = (I - AA^*)^{1/2} \] in \( \mathcal{L}(K) \),
and a defect space \( D_{A^*} = \overline{D_{A^*}(K)} \) contained in \( K \). These operators satisfy the equalities
\[ D_{A^*}A = AD_A \] in \( \mathcal{L}(H,K) \) and \( D_AA^* = A^*D_{A^*} \) in \( \mathcal{L}(K,H) \). (3.2)
The verification of the equalities (3.2) uses the functional calculus for self-adjoint operators in Hilbert space (see [10, Section XXVII.1] for details).

Using the equalities (3.1) and (3.2) in the previous paragraph one verifies that the block operator matrix
\[ \theta_A = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix} : H \oplus D_{A^*} \to K \oplus D_A \]
is a unitary operator. The construction of this unitary operator \( \theta_A \) goes back to Halmos [11].

Let us now return to an \( n \)-hypercontraction \( T \in \mathcal{L}(H) \). We shall need the following lemma.

**Lemma 3.1.** Let \( T \in \mathcal{L}(H) \) be an \( n \)-hypercontraction. Then
\[ \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{*k} T_k = I + \sum_{k=1}^{n-1} D_{k,T}^2 \quad \text{in } \mathcal{L}(H). \]

**Proof.** For \( 1 \leq m \leq n \) we denote by \( \Sigma_m \) the operator
\[ \Sigma_m = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} T^{*k} T^k \quad \text{in } \mathcal{L}(H). \]
Clearly \( \Sigma_1 = I \). For \( m \geq 2 \) we have that
\[ \Sigma_m - \Sigma_{m-1} = (-1)^{m-1} T^{*(m-1)} T^{m-1} + \sum_{k=0}^{m-2} (-1)^k \left( \binom{m}{k+1} - \binom{m-1}{k+1} \right) T^{*k} T^k \quad \text{in } \mathcal{L}(H). \]
We now use the standard formula \( \binom{m}{k+1} = \binom{m-1}{k+1} + \binom{m-1}{k} \) for binomial coefficients to conclude that
\[ \Sigma_m - \Sigma_{m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} T^{*k} T^k = D_{m-1,T}^2 \quad \text{in } \mathcal{L}(H). \]
An easy induction argument now completes the proof of the lemma. □
Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction. We denote by $\mathcal{H}_n$ the Hilbert space $\mathcal{H}$ equipped with the equivalent norm

$$
\|x\|_n^2 = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \|T^k x\|^2 + \sum_{k=1}^{n-1} \|D_{k,T} x\|^2, \quad x \in \mathcal{H}.
$$

(3.3)

The equality of these two expressions for the norm $\|\cdot\|_n$ in (3.3) follows by Lemma 3.1. We denote by $I_n$ the inclusion map of $\mathcal{H}$ into $\mathcal{H}_n$ defined by $I_n x = x$ for $x \in \mathcal{H}$.

**Lemma 3.2.** Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction, and consider the inclusion map $I_n : \mathcal{H} \to \mathcal{H}_n$ defined by $I_n x = x$ for $x \in \mathcal{H}$. Then the adjoint operator $I_n^* \in \mathcal{L}(\mathcal{H}_n, \mathcal{H})$ acts as

$$
I_n^* x = \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^k \right) x, \quad x \in \mathcal{H}_n.
$$

**Proof.** For $x \in \mathcal{H}_n$ and $y \in \mathcal{H}$ we have that

$$
\langle I_n^* x, y \rangle = \langle x, I_n y \rangle_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \langle T^k x, T^k y \rangle = \left\langle \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^k \right\rangle x, y.
$$

This gives the conclusion of the lemma. \qed

We shall consider also the operator $T_n = I_n T$ in $\mathcal{L}(\mathcal{H}, \mathcal{H}_n)$.

**Lemma 3.3.** Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction. Then the operator $T_n = I_n T$ in $\mathcal{L}(\mathcal{H}, \mathcal{H}_n)$ is a contraction operator with defect operator and defect space given by

$$
D_{T_n} = D_{n,T} \quad \text{in } \mathcal{L}(\mathcal{H}) \quad \text{and} \quad D_{T_n} = D_{n,T}.
$$

The adjoint operator $T_n^* \in \mathcal{L}(\mathcal{H}_n, \mathcal{H})$ acts as

$$
T_n^* x = T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^k \right) x, \quad x \in \mathcal{H}_n.
$$

**Proof.** Notice first that

$$
I_n^* I_n = I + \sum_{k=1}^{n-1} D_{k,T}^2 \quad \text{in } \mathcal{L}(\mathcal{H})
$$

by Lemmas 3.1 and 3.2. By the standard formula $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$ for binomial coefficients we have the equalities

$$
D_{k+1,T}^2 = D_{k,T}^2 - T^* D_{k,T}^2 T \quad \text{in } \mathcal{L}(\mathcal{H})
$$
for $1 \leq k \leq n - 1$. Using these equalities we compute that

$$T^*_n T_n = T^* I^*_n I_n T = T^* T + \sum_{k=1}^{n-1} T^* D^2_{k,T} T = T^* T + \sum_{k=1}^{n-1} (D^2_{k,T} - D^2_{k+1,T})$$

$$= T^* T + D^2_{n,T} - D^2_{n,T} = I - D^2_{n,T} \quad \text{in } \mathcal{L}(\mathcal{H}).$$

The equality $T^*_n T_n + D^2_{n,T} = I$ in $\mathcal{L}(\mathcal{H})$ shows that the operator $T_n \in \mathcal{L}(\mathcal{H}, \mathcal{H}_n)$ is a contraction with defect operator and defect space as in the lemma.

The action of the adjoint operator $T^*_n$ is evident by Lemma 3.2. □

We can now refine the description of the wandering subspace from Theorem 2.1.

**Theorem 3.1.** Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction in the class $C_0$. Let the operators $T_n$ and $I_n$ in $\mathcal{L}(\mathcal{H}, \mathcal{H}_n)$ be as above. Then a function $f$ in $A_n(D_{n,T})$ belongs to the wandering subspace $E_{n,T}$ for $I_{n,T}$ if and only if it has the form

$$f = -T^*_n y + S'_n V_n I_n^{-1} D_{n,T} y, \quad y \in D_{T^*_n}.$$ 

Furthermore, we have the norm equality that $\|f\|_{A_n}^2 = \|y\|_n^2$.

**Proof.** By Theorem 2.1 we know that $f \in A_n(D_{n,T})$ belongs to the wandering subspace $E_{n,T}$ for $I_{n,T}$ if and only if it has the form

$$f = a_0 + S'_n V_n x$$

for some elements $a_0 \in D_{n,T}$ and $x \in \mathcal{H}$ such that Eq. (2.2) holds. Using Lemma 3.3 we can rewrite Eq. (2.2) as

$$T^*_n I_n x + D_{T_n} a_0 = 0 \quad (3.4)$$

using the operators $T_n$ and $I_n$. Here $a_0 \in D_{n,T} = D_{T_n}$ and $x \in \mathcal{H}$.

Let us now solve Eq. (3.4). We shall use the unitary operator

$$\theta_{T_n} = \begin{bmatrix} T_n & D_{T_n} \\ D_{T_n}^* & -T^*_n \end{bmatrix} : \mathcal{H} \oplus D_{T_n}^* \rightarrow \mathcal{H}_n \oplus D_{T_n}$$

and its adjoint operator

$$\theta^*_{T_n} = \begin{bmatrix} T_n^* & D_{T_n} \\ D_{T_n}^* & -T_n \end{bmatrix} : \mathcal{H}_n \oplus D_{T_n} \rightarrow \mathcal{H} \oplus D_{T_n}^*.$$ 

Assume now that $x \in \mathcal{H}$ and $a_0 \in D_{T_n}$ satisfies (3.4). Then

$$\theta^*_{T_n} \begin{bmatrix} I_n x \\ a_0 \end{bmatrix} = \begin{bmatrix} T_n^* I_n x + D_{T_n} a_0 \\ D_{T_n}^* I_n x - T_n a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$
where \( y = DT_n I_n x - T_n a_0 \in D_{T_n^*} \). Applying the operator \( \theta T_n \) to this last equality we find that

\[
\begin{bmatrix}
I_n x \\
a_0
\end{bmatrix} = \theta T_n \begin{bmatrix}
0 \\
y
\end{bmatrix} = \begin{bmatrix}
D_{T_n^*} y \\
-T_n^* y
\end{bmatrix}.
\]

This makes evident that every solution \( x \in H \) and \( a_0 \in D_{T_n} \) of Eq. (3.4) is of the form

\[
\begin{align*}
x &= I_n^{-1} D_{T_n^*} y, \\
a_0 &= -T_n^* y
\end{align*}
\]

for some element \( y \in D_{T_n^*} \). Also, if \( x \in H \) and \( a_0 \in D_{T_n} \) are given by (3.5) for some element \( y \in D_{T_n^*} \), then, by property (3.2) of defect operators, Eq. (3.4) holds. We have thus shown that the solutions of (3.4) are parametrized by (3.5). By (3.5) we now have that

\[
f = a_0 + S_n' V_n x = -T_n^* y + S_n' V_n I_n^{-1} D_{T_n^*} y,
\]

where \( y \in D_{T_n^*} \).

Let us now prove the norm equality that \( \| f \|_A^2 = \| y \|_n^2 \). Let \( x \in H \) and \( a_0 \in D_{n,T} \) be given by (3.5). By Theorem 2.2 we have that

\[
\| f \|_A^2 = \| a_0 \|^2 + \sum_{k=0}^{n-1} (-1)^k \left( \begin{array}{c} n \\ k+1 \end{array} \right) \| T^k x \|^2
\]

\[
= \| a_0 \|^2 + \| x \|^2_n + \| T_n^* y \|^2 + \| D_{T_n^*} y \|^2_n = \| y \|_n^2,
\]

where the last equality follows by (3.1). This completes the proof of the theorem. \( \square \)

Let \( T \in \mathcal{L}(H) \) be an \( n \)-hypercontraction. Notice that by Lemma 3.2 the operator \( T_n T_n^* \) in \( \mathcal{L}(H_n) \) acts as

\[
T_n T_n^* x = T T^* \left( \sum_{k=0}^{n-1} (-1)^k \left( \begin{array}{c} n \\ k+1 \end{array} \right) T^{*k} T^k \right) x, \quad x \in H_n.
\]

Since the operator \( T_n \in \mathcal{L}(H, H_n) \) is a contraction by Lemma 3.3, the operator

\[
TT^* \left( \sum_{k=0}^{n-1} (-1)^k \left( \begin{array}{c} n \\ k+1 \end{array} \right) T^{*k} T^k \right) \quad \text{in} \ \mathcal{L}(H)
\]

is self-adjoint in \( \mathcal{L}(H_n) \) and has its spectrum contained in the closed unit interval \([0, 1]\). We denote by \( Q_{n,T} \) the operator

\[
Q_{n,T} = \left( I - TT^* \left( \sum_{k=0}^{n-1} (-1)^k \left( \begin{array}{c} n \\ k+1 \end{array} \right) T^{*k} T^k \right) \right)^{1/2} \quad \text{in} \ \mathcal{L}(H),
\]
where the positive square root is computed in $\mathcal{L}(\mathcal{H}_n)$. We denote by $\mathcal{D}_{n,T}^*$ the closure in $\mathcal{H}$ of the range of the operator $Q_{n,T}$, that is, $\mathcal{D}_{n,T}^* = \overline{Q_{n,T}(\mathcal{H})}$, and we equip this space $\mathcal{D}_{n,T}^*$ with the Hilbert space structure given by the norm $\| \cdot \|_n$ defined by (3.3).

We can now restate Theorem 3.1 using the space $\mathcal{D}_{n,T}^*$ as follows.

**Theorem 3.2.** Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction in the class $C_0$. Then a function $f$ in $A_n(\mathcal{D}_{n,T})$ belongs to the wandering subspace $\mathcal{E}_{n,T}$ for $\mathcal{I}_{n,T}$ if and only if it has the form

$$f = -T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{sk} T^k \right) x + S'_n V_n Q_{n,T} x, \quad x \in \mathcal{D}_{n,T}^*. \quad (3.6)$$

Furthermore, we have the norm equality that $\| f \|_{A_n}^2 = \| x \|_{n}^2$.

**Proof.** Recall the action of the adjoint operator $T^*_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{H})$ given by Lemma 3.3. The map $I_n : \mathcal{H} \to \mathcal{H}_n$ naturally identifies the space $\mathcal{D}_{n,T}^*$ with the defect space $\mathcal{D}_{T_n}^*$. The result is evident by Theorem 3.1. □

We record also the following lemma.

**Lemma 3.4.** Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction. Then

$$T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{sk} T^k \right) Q_{n,T} = D_{n,T} \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{sk} T^k \right)$$

in $\mathcal{L}(\mathcal{H})$.

**Proof.** By (3.2) we have the formula $T^*_n D_{T_n} = D_{T_n} T^*_n$ in $\mathcal{L}(\mathcal{H}_n, \mathcal{H})$. Recall that $D_{T_n} = D_{n,T}$ by Lemma 3.3, and the action of $T^*_n$ given by the same lemma. This makes evident the conclusion of the lemma. □

Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction. We recall from the introduction the definition of the operator-valued analytic function $W_{n,T}$:

$$W_{n,T}(z) = \left[ -T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{sk} T^k \right) + zD_{n,T} \left( \sum_{k=1}^{n} (I - zT)^{-k} \right) Q_{n,T} \right]_{\mathcal{D}_{n,T}^*},$$

$z \in \mathcal{D}$.

By Lemma 3.4 the values $W_{n,T}(z)$ attained by this function $W_{n,T}$ are operators in $\mathcal{L}(\mathcal{D}_{n,T}^* , \mathcal{D}_{n,T})$. Notice that

$$\sum_{k \geq 0} \frac{1}{\mu_{n+k+1}} z^k = \frac{1}{z} \left( \frac{1}{(1-z)^n} - 1 \right) = \sum_{k=1}^{n} \frac{1}{(1-z)^k}, \quad z \in \mathcal{D}. $$
The function \( W_{n,T} \) thus has the power series expansion
\[
W_{n,T}(z) = \left[ -T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{sk} T^k \right) + \sum_{k \geq 1} \frac{1}{\mu_{n;k}} \left( D_{n,T} T^{k-1} Q_{n,T} \right) z^k \right] \bigg|_{D_{n,T}^*},
\]
\( z \in \mathbb{D} \).

(3.7)

We can now parametrize the wandering subspace \( E_{n,T} \) for \( I_{n,T} \) using the function \( W_{n,T} \) as follows.

**Theorem 3.3.** Let \( T \in \mathcal{L}(H) \) be an \( n \)-hypercontraction in the class \( C_0 \). Then a function \( f \) in \( A_n(D_{n,T}) \) belongs to the wandering subspace \( E_{n,T} \) for \( I_{n,T} \) if and only if it has the form
\[
f(z) = W_{n,T}(z)x, \quad z \in \mathbb{D},
\]
for some element \( x \in D_{n,T}^* \). Furthermore, we have the norm equality
\[
\|f\|^2_{A_n} = \|x\|^2_n = \|x\|^2 + \sum_{k=1}^{n-1} \|D_{k,T}x\|^2, \quad x \in D_{n,T}^*,
\]
when \( f \) is of the form (3.8).

**Proof.** Let \( f \in E_{n,T} \) be given by (3.6). By formulas (0.5) and (1.2) we have that
\[
f(z) = -T^* \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} T^{sk} T^k \right) x + \sum_{k \geq 1} \frac{1}{\mu_{n;k}} \left( D_{n,T} T^{k-1} Q_{n,T} \right) z^k, \quad z \in \mathbb{D}.
\]
By the power series expansion (3.7) of the function \( W_{n,T} \) we conclude that
\[
f(z) = W_{n,T}(z)x, \quad z \in \mathbb{D}.
\]
The conclusion of the theorem is now evident by Theorem 3.2. \( \square \)

We remark that in the case \( n = 1 \) of a contraction operator \( T \in \mathcal{L}(H) \) the \( \mathcal{L}(D_{T^*}, D_T) \)-valued analytic function
\[
W_T(z) = W_{1,T}(z) = [\left( -T^* + zD_T(I - zT)^{-1}D_{T^*} \right)]_{D_{T^*}}, \quad z \in \mathbb{D},
\]
is the characteristic operator function studied by Sz.-Nagy and Foias (see [26, Chapter VI]).

**4. Multiplier properties of the function \( W_{n,T} \)**

In this section we discuss some multiplier properties of the function \( W_{n,T} \). We first show that the function \( W_{n,T} \) acts as a contractive multiplier from the Hardy space \( A_1(D_{n,T}^*) \) into the Bergman space \( A_n(D_{n,T}) \).
Theorem 4.1. Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction in the class $C_0$. Then the function $W_{n,T}$ acts as a contractive multiplier $W_{n,T} : f \mapsto W_{n,T}f$ from the Hardy space $A_1(D^*_n,T)$ into the Bergman space $A_n(D_n,T)$:

$$\|W_{n,T}f\|_{A_n}^2 \leq \|f\|_{A_1}^2, \quad f \in A_1(D^*_n,T);$$

here the space $D^*_n,T$ is equipped with the norm $\| \cdot \|_n$ given by (3.3).

Proof. Let $f \in A_1(D^*_n,T)$ be a polynomial of the form (0.1) with coefficients $a_k \in D^*_n,T$. We write the function $W_{n,T}f$ as

$$W_{n,T}f = \sum_{k \geq 0} S_n^k W_{n,T}a_k.$$

Recall that by Theorem 3.3 the elements $W_{n,T}a_k$ all belong to the wandering subspace $\mathcal{E}_{n,T}$. We rewrite the sum for $W_{n,T}f$ as

$$f = W_{n,T}a_0 + S_n \left( \sum_{k \geq 0} S_n^k W_{n,T}a_{k+1} \right).$$

Now, since the wandering subspace $\mathcal{E}_{n,T}$ for $\mathcal{I}_{n,T}$ is orthogonal to $S_n(\mathcal{I}_{n,T})$, we have that

$$\|W_{n,T}f\|_{A_n}^2 = \|W_{n,T}a_0\|_{A_n}^2 + \left\| S_n \left( \sum_{k \geq 0} S_n^k W_{n,T}a_{k+1} \right) \right\|_{A_n}^2$$

$$= \|a_0\|_n^2 + \left\| S_n \left( \sum_{k \geq 0} S_n^k W_{n,T}a_{k+1} \right) \right\|_{A_n}^2,$$

where the last equality follows by Theorem 3.3. The fact that the shift operator $S_n$ on $A_n(D_n,T)$ is a contraction now gives that

$$\|W_{n,T}f\|_{A_n}^2 \leq \|a_0\|_n^2 + \left\| \sum_{k \geq 0} S_n^k W_{n,T}a_{k+1} \right\|_{A_n}^2. \quad (4.1)$$

We can now iterate this last inequality (4.1) to obtain that

$$\|W_{n,T}f\|_{A_n}^2 \leq \sum_{k \geq 0} \|a_k\|_n^2.$$

Since the space of $D^*_n,T$-valued polynomials is dense in $A_1(D^*_n,T)$, an approximation argument now yields the conclusion of the theorem. $\square$

Remark 4.1. We remark that the closure in $A_n(D_n,T)$ of the range of the multiplier $W_{n,T} : A_1(D^*_n,T) \to A_n(D_n,T)$, that is, the closure in $A_n(D_n,T)$ of the set of all functions of the form $W_{n,T}g$, where $g \in A_1(D^*_n,T)$ is a $D^*_n,T$-valued polynomial, equals the shift invariant subspace $[\mathcal{E}_{n,T}]$ generated by the wandering subspace $\mathcal{E}_{n,T}$. In particular, the multiplier $W_{n,T}$ maps $A_1(D^*_n,T)$ into $\mathcal{I}_{n,T}$.
We recall that the space $D^*_{n,T}$ is equipped with the norm $\| \cdot \|_n$ given by (3.3). In particular, this means that the norm of $A_1(D^*_{n,T})$ is given by

$$\| f \|_{A_1}^2 = \sum_{k \geq 0} \| a_k \|_n^2$$

for $f \in A_1(D^*_{n,T})$ as in (0.1).

Let us consider the case $n = 1$ in some more detail.

**Corollary 4.1.** Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction in the class $C_0$. Then the characteristic operator function $W_T = W_{1,T}$ is an isometric multiplier $W_T : f \mapsto W_T f$ from the Hardy space $A_1(D^*_T)$ into the Hardy space $A_1(D_T)$ with range equal to $I_{1,T}$.

**Proof.** In this case the shift operator $S_1$ on $A_1(D_T)$ is an isometry and we have equality in (4.1). This gives that the multiplier $W_T$ maps $A_1(D^*_T)$ isometrically into $A_1(D_T)$. By the von Neumann–Wold decomposition of an isometry (see [26, Section I.1]), the range of the multiplier $W_T$ equals $I_{1,T}$ (see Remark 4.1). \qed

We remark that the proof of Theorem 4.1 is modeled on an argument of Shimorin [25, Lemma 2.1].

**Remark 4.2.** In the scalar case when the defect spaces $D_{n,T}$ and $D^*_{n,T}$ are both one-dimensional and $n = 2$ the result of Theorem 4.1 is due to Hedenmalm [13,15]. The case $n = 3$ goes back to Hedenmalm [14].

We next show that the multiplier $W_{n,T} : A_1(D^*_{n,T}) \to A_n(D_{n,T})$ is injective.

**Proposition 4.1.** Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction in the class $C_0$. Let the function $W_{n,T}$ act as a multiplier from $A_1(D^*_{n,T})$ into $A_n(D_{n,T})$ as in Theorem 4.1. Denote by $L$ the operator

$$L = \left( (S_n|_{I_{n,T}})^* S_n|_{I_{n,T}} \right)^{-1} (S_n|_{I_{n,T}})^* \quad \text{in } \mathcal{L}(I_{n,T}).$$

Then the intertwining relations

$$S_n W_{n,T} = W_{n,T} S_1 \quad \text{and} \quad LW_{n,T} = W_{n,T} L_1$$

holds. In particular, the multiplier $W_{n,T} : A_1(D^*_{n,T}) \to A_n(D_{n,T})$ is injective.

**Proof.** The first intertwining relation $S_n W_{n,T} = W_{n,T} S_1$ is obvious. Let us verify the second intertwining relation $LW_{n,T} = W_{n,T} L_1$. Recall from Section 1 that the operator $L$ in $\mathcal{L}(I_{n,T})$ is the left-inverse of $S_n|_{I_{n,T}}$ with kernel $\ker L = \ker (S_n|_{I_{n,T}})^* = \mathcal{E}_{n,T}$. Let

$$g(z) = \sum_{k \geq 0} b_k z^k, \quad z \in \mathbb{D},$$

(4.2)
be a $\mathcal{D}_{n,T}^*$-valued polynomial. The function $f = W_{n,T}g$ has the form

$$f = \sum_{k \geq 0} S_n^k W_{n,T} b_k$$

and the elements $W_{n,T} b_k$ all belong to $\mathcal{E}_{n,T}$ (see Theorem 3.3). We now have that

$$L f = \sum_{k \geq 1} S_n^{k-1} W_{n,T} b_k = \sum_{k \geq 0} S_n^k W_{n,T} b_{k+1} = W_{n,T} L g.$$

This shows that $L W_{n,T} g = W_{n,T} L g$ when $g$ is a $\mathcal{D}_{n,T}^*$-valued polynomial. The intertwining relation $L W_{n,T} = W_{n,T} L$ now follows by a standard approximation argument.

Let us now prove that the multiplier $W_{n,T}: A_1(\mathcal{D}_{n,T}^*) \to A_n(\mathcal{D}_{n,T})$ is injective. We shall use the operator $P = I - S_n L$ in $L(I_{n,T})$ which is the orthogonal projection of $I_{n,T}$ onto $\mathcal{E}_{n,T}$ (see Section 1). Let $f = W_{n,T}g$, where $g \in A_1(\mathcal{D}_{n,T}^*)$ is given by (4.2). A computation using the intertwining relations shows that

$$P L^k f = (I - S_n L)L^k W_{n,T} g = W_{n,T}(I - S_1 L_1)L^k g = W_{n,T} b_k, \quad k \geq 0.$$

By Theorem 3.3 this last equality determines the coefficients $b_k$ uniquely. This completes the proof of the proposition. □

Recall that the reproducing kernel for a Hilbert space $\mathcal{H}$ of $\mathcal{E}$-valued analytic functions in the unit disc $\mathbb{D}$ is the function $K_{\mathcal{H}}: \mathbb{D} \times \mathbb{D} \to L(\mathcal{E})$ satisfying the conditions that $K_{\mathcal{H}}(\cdot, \zeta)x$ belongs to $\mathcal{H}$ for every $\zeta \in \mathbb{D}$ and $x \in \mathcal{E}$, and that

$$\langle f(\zeta), x \rangle = \langle f, K_{\mathcal{H}}(\cdot, \zeta)x \rangle_{\mathcal{H}}, \quad \zeta \in \mathbb{D}, \ f \in \mathcal{H},$$

for $x \in \mathcal{E}$. This last property (4.3) is called the reproducing property of the kernel function $K_{\mathcal{H}}$. We remind that the Bergman space $A_n(\mathcal{E})$ has the reproducing kernel

$$K_n(z, \zeta) = \frac{1}{(1 - \overline{\zeta}z)^n} I_{\mathcal{E}}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D},$$

where $I_{\mathcal{E}}$ denotes the identity operator on $\mathcal{E}$.

We next compute the reproducing kernel for the space $V_n(\mathcal{H})$.

**Proposition 4.2.** Let $T \in \mathcal{L}(\mathcal{H})$ be an $n$-hypercontraction in the class $C_0$. Then the reproducing kernel for the space $V_n(\mathcal{H})$ is the $\mathcal{D}_{n,T}$-valued function given by

$$K_{V_n(\mathcal{H})}(z, \zeta) = D_{n,T}(I - zT)^{-n} (I - \overline{\zeta}T^*)^{-n} D_{n,T}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$

**Proof.** Let $f = V_n x$ be a function in $V_n(\mathcal{H})$. Then for $y \in D_{n,T}$ we have that

$$\langle f(\zeta), y \rangle = \langle D_{n,T}(I - \zeta T)^{-n} x, y \rangle = \langle x, (I - \overline{\zeta} T^*)^{-n} D_{n,T} y \rangle.$$
The fact that the operator \( V_n \) in \( \mathcal{L}(\mathcal{H}, A_n(D_n,T)) \) is an isometry now gives that
\[
\langle f(\zeta), y \rangle = \langle V_n x, V_n (I - \bar{\zeta} T^*)^{-n} D_{n,T} y \rangle_{A_n} = \langle f, V_n (I - \bar{\zeta} T^*)^{-n} D_{n,T} y \rangle_{A_n}.
\]
This completes the proof of the proposition.

We denote by \( W \) the range of the multiplier \( W_{n,T} : g \mapsto W_{n,T} g \) mapping \( A_1(D_{n,T}^*) \) into \( A_n(D_{n,T}) \) by Theorem 4.1, that is, the space \( W \) consists of all functions \( f \in A_n(D_{n,T}) \) of the form
\[
f(z) = W_{n,T}(z)g(z), \quad z \in D,
\]
for some \( g \in A_1(D_{n,T}^*) \). Recall that by Proposition 4.1 the function \( f \in W \) uniquely determines the function \( g \in A_1(D_{n,T}^*) \) by (4.5). We equip the space \( W \) with the norm induced by \( A_1(D_{n,T}^*) \), that is, we set \( \| f \|^2_W = \| g \|^2_{A_1} \) when \( f \in W \) and \( g \in A_1(D_{n,T}^*) \) are related as in (4.5). In this way the space \( W \) becomes a Hilbert space of \( D_{n,T} \)-valued analytic functions in \( D \). We next compute the reproducing kernel function for the space \( W \).

**Lemma 4.1.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be an \( n \)-hypercontraction in the class \( C_0 \). Let the Hilbert space \( W \) of \( D_{n,T} \)-valued analytic functions in \( D \) be defined as in the previous paragraph. Then the reproducing kernel for the space \( W \) is given by
\[
K_W(z, \zeta) = \frac{1}{1 - \bar{\zeta} z} W_{n,T}(z) W_{n,T}(\zeta)^*, \quad (z, \zeta) \in D \times D.
\]
**Proof.** Let \( f \in W \) and \( g \in A_1(D_{n,T}^*) \) be as in (4.5). For \( x \in D_{n,T} \) we have that
\[
\langle f(\zeta), x \rangle = \langle W_{n,T}(\zeta)g(\zeta), x \rangle = \langle g(\zeta), W_{n,T}(\zeta)^* x \rangle_{A_1}.
\]
By the reproducing property of the kernel function \( K_1 \) for the space \( A_1(D_{n,T}^*) \) we have that
\[
\langle f(\zeta), x \rangle = \langle g, K_1(\cdot, \zeta) W_{n,T}(\zeta)^* x \rangle_{A_1}.
\]
Now since the function \( W_{n,T} \) acts as an isometric multiplier of \( A_1(D_{n,T}^*) \) onto the space \( W \) we conclude that
\[
\langle f(\zeta), x \rangle = \langle W_{n,T} g, W_{n,T} K_1(\cdot, \zeta) W_{n,T}(\zeta)^* x \rangle_W = \langle f, W_{n,T} K_1(\cdot, \zeta) W_{n,T}(\zeta)^* x \rangle_W.
\]
This completes the proof of the lemma.

The contractive multiplier property from Theorem 4.1 leads to the following result on domination of reproducing kernel functions.

**Theorem 4.2.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be an \( n \)-hypercontraction in the class \( C_0 \). Then the \( D_{n,T} \)-valued function
\[ L(z, \zeta) = \frac{1}{(1 - \bar{\zeta}z)^n} I_{D_n,T} - D_{n,T}(I - zT)^{-n}(I - \bar{T}^*)^{-n} D_{n,T} \]
\[ - \frac{1}{1 - \bar{\zeta}z} W_{n,T}(z) W_{n,T}(\zeta)^*, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}, \]

is positive definite on \( \mathbb{D} \times \mathbb{D} \). In particular, we have the inequality
\[ \frac{1}{1 - |z|^2} W_{n,T}(z) W_{n,T}(z)^* + D_{n,T}(I - zT)^{-n}(I - \bar{T}^*)^{-n} D_{n,T} \]
\[ \leq \frac{1}{(1 - |z|^2)^n} I_{D_n,T} \quad \text{in} \ \mathcal{L}(D_n,T), \ z \in \mathbb{D}. \]

**Proof.** Let the space \( \mathcal{W} \) be as in Lemma 4.1. By Theorem 4.1 and Remark 4.1 the space \( \mathcal{W} \) is contractively embedded into \( \mathcal{I}_{n,T} \). By this we mean that \( \mathcal{W} \subset \mathcal{I}_{n,T} \) and \( \|f\|_{A_n} \leq \|f\|_{\mathcal{Y}} \) for \( f \in \mathcal{W} \). Recall that the space \( A_n(D_{n,T}) \) is the orthogonal sum of the subspaces \( V_n(\mathcal{H}) \) and \( \mathcal{I}_{n,T} \). The reproducing kernel function for the space \( \mathcal{I}_{n,T} \) is given by
\[ K_{\mathcal{I}_{n,T}}(z, \zeta) = K_n(z, \zeta) - K_{V_n(\mathcal{H})}(z, \zeta) \]
\[ = \frac{1}{(1 - \bar{\zeta}z)^n} I_{D_n,T} - D_{n,T}(I - zT)^{-n}(I - \bar{T}^*)^{-n} D_{n,T}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}, \]
where the last equality follows by Proposition 4.2 and (4.4). The reproducing kernel function \( K_{\mathcal{W}} \) for the space \( \mathcal{W} \) was computed in Lemma 4.1. It is known that a contractive embedding \( \mathcal{W} \subset \mathcal{I}_{n,T} \) is equivalent to the domination relation \( K_{\mathcal{W}} \ll K_{\mathcal{I}_{n,T}} \) of reproducing kernel functions (see [7, Section I.7]). We conclude that the function
\[ L(z, \zeta) = K_{\mathcal{I}_{n,T}}(z, \zeta) - K_{\mathcal{W}}(z, \zeta), \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}, \]
is positive definite on \( \mathbb{D} \times \mathbb{D} \). This gives the positive definiteness assertion in the theorem. The last inequality in the theorem follows by setting \( z = \zeta \) noticing that \( L(z, z) \geq 0 \) in \( \mathcal{L}(D_{n,T}) \) by positive definiteness of the function \( L \). This completes the proof of the theorem. \( \Box \)

In the case \( n = 2 \) of scalar-valued Bergman inner functions the inequality in Theorem 4.2 seems first to have appeared in Zhu [27, Theorem 4.2].

Let us examine the case \( n = 1 \) somewhat closer.

**Corollary 4.2.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be a contraction in the class \( C_0 \). Then
\[ \frac{1}{1 - \bar{\zeta}z} I_{D_T} = \frac{1}{1 - \bar{\zeta}z} W_T(z) W_T(\zeta)^* + D_T(I - zT)^{-1}(I - \bar{T}^*)^{-1} D_T, \]
\[ (z, \zeta) \in \mathbb{D} \times \mathbb{D}. \]

**Proof.** The space \( A_1(D_{T^*}) \) is the orthogonal sum of the subspaces \( V_1(\mathcal{H}) \) and \( \mathcal{I}_{1,T} \). By Corollary 4.1 we know that the characteristic operator function \( W_T \) is an isometric multiplier from \( A_1(D_{T^*}) \) onto \( \mathcal{I}_{1,T} \). The reproducing kernel functions decompose accordingly. \( \Box \)
We remark that the result of Corollary 4.2 is a well-known formula in the context of unitary systems which can be proved by direct computation (see [10, Section XXVIII.2]). We have included it here merely as an illustration of our methods.

5. Shift invariant subspaces in Bergman spaces

The considerations in the previous sections have some consequences concerning general shift invariant subspaces in Bergman spaces. Let \( \mathcal{H} \) be a shift invariant subspace of \( A_n(\mathcal{E}) \). We set

\[ \mathcal{H} = A_n(\mathcal{E}) \ominus \mathcal{I} \quad \text{and} \quad T = S_n^*|_\mathcal{H}. \]

The operator \( T \) in \( \mathcal{L}(\mathcal{H}) \) is an \( n \)-hypercontraction in the class \( C_0 \). We can now model this operator \( T \) as part of the adjoint shift operator \( S_n^* \) on the space \( A_n(\mathcal{D}_{n,T}) \) by means of the formula

\[
(V_n f)(z) = D_{n,T}(I - zT)^{-n} f = \sum_{k \geq 0} \binom{k + n - 1}{k} (D_{n,T} T^k f) z^k, \quad z \in \mathbb{D}.
\]

Furthermore, by a uniqueness property of this operator model there exists an isometry \( \hat{V}_n : \mathcal{D}_{n,T} \to \mathcal{E} \) such that the functions \( f \in \mathcal{H} \) all admit the representation

\[
f(z) = \hat{V}_n((V_n f)(z)), \quad z \in \mathbb{D}.
\]  

(5.1)

The isometry \( \hat{V}_n : \mathcal{D}_{n,T} \to \mathcal{E} \) of coefficient spaces is uniquely determined by (5.1) and acts as

\[
\hat{V}_n : \mathcal{D}_{n,T} f \mapsto f(0) \quad \text{for} \quad f \in \mathcal{H}.
\]

Full details of this construction can be found in [22, Sections 6 and 7].

We write \( \hat{\mathcal{E}} = \hat{V}_n(\mathcal{D}_{n,T}) \subset \mathcal{E} \). The map \( \hat{V}_n \) naturally extends to an isometry of \( A_n(\mathcal{D}_{n,T}) \) into \( A_n(\mathcal{E}) \) with range equal to \( A_n(\hat{\mathcal{E}}) \) by setting

\[
(\hat{V}_n f)(z) = \hat{V}_n(f(z)), \quad z \in \mathbb{D},
\]

for \( f \in A_n(\mathcal{D}_{n,T}) \).

We have the following description of a general shift invariant subspace of \( A_n(\mathcal{E}) \).

**Theorem 5.1.** Let \( \mathcal{I} \) be a shift invariant subspace of \( A_n(\mathcal{E}) \), and let \( \mathcal{H} = A_n(\mathcal{E}) \ominus \mathcal{I} \) and \( T = S_n^*|_\mathcal{H} \) in \( \mathcal{L}(\mathcal{H}) \) be as above. Then the space \( \mathcal{I} \) decomposes as an orthogonal sum

\[
\mathcal{I} = A_n(\mathcal{E} \ominus \hat{\mathcal{E}}) \oplus \hat{V}_n(\mathcal{I}_{n,T}),
\]

that is, a function \( f \in A_n(\mathcal{E}) \) belongs to \( \mathcal{I} \) if and only if it has the form of an orthogonal sum \( f = f_1 + \hat{V}_n g \), where \( f_1 \in A_n(\mathcal{E}) \) has all its Taylor coefficients in \( \mathcal{E} \ominus \hat{\mathcal{E}} \) and \( g \) belongs to the shift invariant subspace \( \mathcal{I}_{n,T} \) of \( A_n(\mathcal{D}_{n,T}) \).

**Proof.** By formula (5.1) we have that \( \hat{V}_n(V_n(\mathcal{H})) = \mathcal{H} \). In particular, the space \( \mathcal{H} \) is contained already in \( A_n(\hat{\mathcal{E}}) \). Passing to the orthogonal complement we have that
This completes the proof of the theorem. □

Recall that the wandering subspace \( E_I \) for a shift invariant subspace \( I \) of \( A_n(\mathcal{E}) \) is the subspace

\[
E_I = I \ominus \mathcal{S}_n(I)
\]
of \( I \). We have the following description of a general wandering subspace.

**Theorem 5.2.** Let \( I \) be a shift invariant subspace of \( A_n(\mathcal{E}) \), and let \( \mathcal{H} = A_n(\mathcal{E}) \ominus I \) and \( T = S_n^*|_{\mathcal{H}} \) in \( \mathcal{L}(\mathcal{H}) \) be as above. Then a function \( f \in A_n(\mathcal{E}) \) belongs to the wandering subspace \( \mathcal{E}_I \) for \( I \) if and only if it has the form

\[
f(z) = a_0 + \hat{V}_n W_{n,T}(z) g, \quad z \in \mathbb{D},
\]

where \( a_0 \in \mathcal{E} \ominus \hat{\mathcal{E}} \) and \( g \) belongs to the defect space \( \mathcal{D}_{n,T}^* \). The norm of a function \( f \in A_n(\mathcal{E}) \) of the form (5.2) is given by

\[
\| f \|_{A_n}^2 = \|a_0\|^2 + \|g\|_{n}^2 = \|a_0\|^2 + \sum_{k \geq 0} \frac{\mu_{n,k}^2}{\mu_{n,k+1}} \|b_k\|^2,
\]

where \( b_k \) is the \( k \)th Taylor coefficient of \( g \) as in (4.2).

**Proof.** The form of the invariant subspace \( I \) was calculated in Theorem 5.1. By this description we have that

\[
\mathcal{E}_I = (\mathcal{E} \ominus \hat{\mathcal{E}}) \ominus \hat{V}_n(\mathcal{E}_{n,T}),
\]

where \( \mathcal{E}_{n,T} \) is the wandering subspace for the shift invariant subspace \( I_{n,T} \) of \( A_n(\mathcal{D}_{n,T}) \). The wandering subspace \( \mathcal{E}_{n,T} \) was described in Theorem 3.3 as the space of all functions of the form

\[
f(z) = W_{n,T}(z) g, \quad z \in \mathbb{D},
\]

where \( g \) belongs to \( \mathcal{D}_{n,T}^* \), and norm given by \( \| f \|_{A_n}^2 = \|g\|_{n}^2 \). The expression for the norm \( \| \cdot \|_{n} \) using Taylor coefficients follows by Proposition 1.1. This yields the conclusion of the theorem. □

We have the following characterization of a related class of operator-valued analytic functions.

**Theorem 5.3.** Let \( W \) be an \( \mathcal{L}(\mathcal{F}, \mathcal{E}) \)-valued analytic function in the unit disc \( \mathbb{D} \) such that for every \( x \in \mathcal{F} \) the function \( Wx : z \mapsto W(z)x \) belongs to \( A_n(\mathcal{E}) \) with the properties that
• the norm equality \( \|Wx\|_{A_n}^2 = \|x\|_2^2 \) holds for every \( x \in \mathcal{F} \), and
• \( Wx \perp S^k_n Wx \) for all \( k \geq 1 \) and \( x \in \mathcal{F} \).

Then \( W(\mathcal{F}) = \{Wx: x \in \mathcal{F}\} \) is a wandering subspace in \( A_n(\mathcal{E}) \). Denote by \( \mathcal{I} \) the shift invariant subspace generated by \( W(\mathcal{F}) \) in \( A_n(\mathcal{E}) \), that is, \( \mathcal{I} = [W(\mathcal{F})] \). Let \( \mathcal{H} = A_n(\mathcal{E}) \ominus \mathcal{I} \) and \( T = S^1_n|\mathcal{H} \in \mathcal{L}(\mathcal{H}) \). Then there exists a unitary operator

\[
U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} : \mathcal{F} \to (\mathcal{E} \ominus \hat{\mathcal{E}}) \oplus D^*_n,T
\]

of \( \mathcal{F} \) onto \( (\mathcal{E} \ominus \hat{\mathcal{E}}) \oplus D^*_n,T \) such that

\[
W(z)x = U_1x + \hat{V}_n W_{n,T}(z)U_2x, \quad x \in \mathcal{F}, \; z \in \mathbb{D}.
\]

Furthermore, the equality (5.3) determines the operators \( U_1 \) and \( U_2 \) uniquely.

**Proof.** Clearly \( W(\mathcal{F}) \) is a closed subspace of \( A_n(\mathcal{E}) \). By polarization we have that \( Wx \perp S^k_n Wy \) for \( k \geq 1 \) and \( x, y \in \mathcal{F} \). This shows that \( W(\mathcal{F}) \) satisfies the defining property of a wandering subspace, that is, \( W(\mathcal{F}) \perp S^k_n W(\mathcal{F}) \) for \( k \geq 1 \).

We have that \( \mathcal{E}_\mathcal{I} = W(\mathcal{F}) \) (see Remark 5.1). By Theorem 5.2 the wandering subspace \( \mathcal{E}_\mathcal{I} = W(\mathcal{F}) \) for \( \mathcal{I} \) consists of all functions \( f \) in \( A_n(\mathcal{E}) \) of the form (5.2) with norm equality

\[
\|f\|_{A_n}^2 = \|a_0\|^2 + \|g\|_n^2.
\]

For \( x \in \mathcal{F} \) we set \( Ux = (a_0, g) \) when \( f = Wx \) is given by (5.2). This gives us a unitary map \( U \) from \( \mathcal{F} \) onto \( (\mathcal{E} \ominus \hat{\mathcal{E}}) \oplus D^*_n,T \). The operators \( U_1 \) and \( U_2 \) are uniquely determined by (5.3) by uniqueness of the representation (5.2). This completes the proof of the theorem. □

**Remark 5.1.** It is a general fact often accredited to Halmos [12] that a wandering subspace is uniquely determined by the invariant subspace it generates. Let \( \mathcal{T} \in \mathcal{L}(\mathcal{H}) \) be a bounded linear operator, and let \( \mathcal{E} \) be a closed subspace of \( \mathcal{H} \) such that \( \mathcal{E} \perp T^k(\mathcal{E}) \) for \( k \geq 1 \). Set \( \mathcal{I} = [\mathcal{E}] = \bigvee_{k \geq 0} T^k(\mathcal{E}) \). Then \( \mathcal{I} = \mathcal{E} \ominus (\bigvee_{k \geq 1} T^k(\mathcal{E})) \), which gives that \( \mathcal{E}_\mathcal{I} = \mathcal{I} \ominus T(\mathcal{I}) = \mathcal{E} \).

We recall that the operator-valued analytic function \( W_{n,T} \) is a contractive multiplier from the Hardy space \( A_1(D^*_n,T) \) into the Bergman space \( A_n(D,n,T) \) (see Theorem 4.1), and that a related upper bound is available (see Theorem 4.2).

Specializing to the case \( n = 1 \) we obtain a parametrization of the shift invariant subspaces of the Hardy space \( A_1(\mathcal{E}) \) for a general not necessarily separable Hilbert space \( \mathcal{E} \).

**Corollary 5.1.** Let \( \mathcal{I} \) be a shift invariant subspace of the Hardy space \( A_1(\mathcal{E}) \). Then a function \( f \) in \( A_1(\mathcal{E}) \) belongs to the subspace \( \mathcal{I} \) if and only if it has the form

\[
f(z) = f_1(z) + \hat{V}_1 W_T(z)g(z), \quad z \in \mathbb{D}, \tag{5.4}
\]

for some functions \( f_1 \in A_1(\mathcal{E} \ominus \hat{\mathcal{E}}) \) and \( g \in A_1(D^*_T) \). Furthermore, we have the norm equality \( \|f\|_{A_1}^2 = \|f_1\|_{A_1}^2 + \|g\|_{A_1}^2 \) when \( f \in \mathcal{I} \), \( f_1 \in A_1(\mathcal{E} \ominus \hat{\mathcal{E}}) \) and \( g \in A_1(D^*_T) \) are related by (5.4).
Proof. Recall the form of an invariant subspace calculated in Theorem 5.1. By Corollary 4.1 the characteristic operator function $W_T$ is an isometric multiplier from the Hardy space $A_1(D_T^*)$ into the Hardy space $A_1(D_T)$ with range equal to $I_{1,T}$. This completes the proof of the corollary. 

The previous results yield the following consequence concerning the index $\dim E_I$ of a shift invariant $I$ of $A_n(E)$.

Corollary 5.2. Let $I$ be a shift invariant subspace of $A_n(E)$ with $E$ separable. Set $\mathcal{H} = A_n(E) \ominus I$ and $T = S_n|_{\mathcal{H}}$. Then $\dim D_{n,T} \leq \dim E$. If the defect index $\dim D_{n,T}$ is finite, then

$$\dim E_I = \dim E - \dim D_{n,T} + \dim D_{n,T}^*.$$ 

Proof. The first inequality $\dim D_{n,T} \leq \dim E$ is evident by the fact that the operator $\hat{V}_n \in \mathcal{L}(D_{n,T}, E)$ is an isometry (see [22, Section 7]). The second inequality is evident by the description of the wandering subspace $E_I$ in Theorem 5.2. 

We notice also that $\dim E_I = \dim D_{n,T}^*$ if $\hat{V}_n(D_{n,T}) = E$.

It has been known for some time that even in the scalar case $E = \mathbb{C}$ the index $\dim E_I$ of a shift invariant subspace $I$ of $A_1(E)$ for $n \geq 2$ can equal any positive integer or $+\infty$. This was first proved by Apostol et al. [5] using dual algebras, and later more explicit constructions have been found by Hedenmalm et al. [19] and others.

In the context of the Hardy space $A_1(E)$ with $E$ separable it is a result of Halmos [12] that the index $\dim E_I$ of a shift invariant subspace $I$ of $A_1(E)$ cannot exceed the index of the whole space $A_1(E)$ meaning that $\dim E_I \leq \dim E$. This inequality is naturally interpreted as an inequality of defect indexes as follows.

Proposition 5.1. Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction operator in the class $C_0$, acting on a separable Hilbert space $\mathcal{H}$. Then $\dim D_T^* \leq \dim D_T$.

Proof. It is known that the characteristic operator function $W_T$ has non-tangential boundary values $W_T(e^{i\theta})$ in the strong operator topology for a.e. $e^{i\theta} \in \mathbb{T}$. A well-known argument then shows that the operator $W_T(e^{i\theta})$ is an isometry in $\mathcal{L}(D_T^*, D_T)$ for a.e. $e^{i\theta} \in \mathbb{T}$ (see [26, Chapter V]). This gives the conclusion of the proposition. 

For an $n$-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ in the class $C_0$, and $n \geq 2$ the corresponding inequality $\dim D_{n,T}^* \leq \dim D_{n,T}$ of defect indexes is not true in general for the reasons quoted above.

References