AN INEQUALITY FOR SUMS OF SUBHARMONIC AND SUPERHARMONIC FUNCTIONS

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Abstract. We prove a general inequality for the distributional Laplacian of a sum of a subharmonic and a superharmonic function postcomposed with a convex function of linear growth. We use this inequality to show that convex functions of linear growth operate by means of postcomposition on the class of sums of subharmonic and superharmonic functions.

0. Introduction

Let \( u \) be a sum of a subharmonic and a superharmonic function in an open set \( \Omega \) in \( \mathbb{R}^n, n \geq 2 \), and let \( f : \mathbb{R} \to \mathbb{R} \) be a convex function of linear growth. In this paper we consider the problem of finding lower bounds for the distributional Laplacian \( \Delta(f \circ u) \) of the composite function \( f \circ u \) in terms of the Laplacian \( \Delta u \). The principal result of this paper, Theorem 1, is an estimate of this type.

Estimates of Riesz mass of the type considered in this paper goes back to work of Tosio Kato [7]. The topic has recently been considered by Haïm Brezis and Augusto C. Ponce [1] and independently by the author (see [8, 9] where some results were announced but no proofs given). Compared to [1, 7] the new contribution in this paper is that we allow \( f \) to be a general convex function whereas in these papers only the cases \( f(x) = x^+ = \max(x, 0) \) or \( f(x) = |x| \) were considered.

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Following Brezis and Ponce [1] we formulate our results in terms of Lebesgue decompositions with respect to capacity, that is, Newtonian or logarithmic capacity depending on the dimension \( n \geq 2 \). Notably we show, generalizing a result from [1], that when restricted to the singular with respect to capacity parts equality holds in our main estimate (see Theorem 4).

Using Theorem 1 we show that convex functions \( f : \mathbb{R} \to \mathbb{R} \) of linear growth operate on the class of sums of subharmonic and superharmonic functions, that is, if \( u \) is a sum of a subharmonic and a superharmonic function then the composite function \( f \circ u \) is also a function of this type (see Theorem 2).

Specializing to subharmonic functions we show that

\[
\Delta(f \circ u) \geq f'(u+)\Delta u \quad \text{in } \mathcal{D}'(\Omega)
\]

whenever \( u \) is subharmonic in \( \Omega \) and \( f : \mathbb{R} \to \mathbb{R} \) is convex and increasing (see Theorem 3). This last inequality can be considered as a strengthened form of the well-known fact that increasing convex functions operate on the class of subharmonic functions (see [5, Theorem 2.2]).

The exact statements of our results are given in Section 1. The proof of Theorem 1 is given in Section 2. This proof is self-contained and depends on careful regularization arguments. In Section 3 we compare our results with those of [1, 7].

We wish to mention that estimates of Riesz mass of the type considered in this paper has proved to be useful in applications to partial differential equations (see [2, 7, 8, 9]).

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Notation and terminology. The Euclidean ball with center \( x \) and radius \( r > 0 \) is denoted by \( B(x, r) \). By \( \Omega \) we will always mean an open set in \( \mathbb{R}^n \), where \( n \geq 2 \). A subharmonic function \( u \) in \( \Omega \) is a not identically \( -\infty \) upper semicontinuous function \( u : \Omega \to [-\infty, \infty) \) satisfying the usual mean value inequalities. A function \( u \) is said to be superharmonic if \(-u \) is subharmonic. By capacity we mean Newtonian or logarithmic capacity depending on the dimension \( n \geq 2 \). The capacity of a Borel set \( E \) is denoted by \( \text{cap}(E) \). As a standard reference for subharmonic functions and capacity we use [5]. The space of distributions in \( \Omega \) is denoted by \( \mathcal{D}'(\Omega) \) and by \( \mathcal{D}(\Omega) \) we denote the space of all compactly supported \( C^\infty \)-smooth test functions in \( \Omega \). For \( u, v \in \mathcal{D}'(\Omega) \), the inequality \( u \leq v \) in \( \mathcal{D}'(\Omega) \) means that \( \langle u, \varphi \rangle \leq \langle v, \varphi \rangle \) for all \( 0 \leq \varphi \in \mathcal{D}(\Omega) \). A distribution of order 0 is referred to as a Radon measure. A standard reference for distribution theory is [6].
1. Statement of results

We shall consider functions \( u \) in \( \Omega \) that are locally sums \( u = v + w \) of a subharmonic function \( v \) and a superharmonic function \( w \). In the literature such a function \( u \) is sometimes called \( \delta \)-subharmonic. For such a sum decomposition \( u = v + w \) we have an exceptional set \( E \) consisting of those points \( x \in \Omega \) where \( v(x) = -\infty \) or \( w(x) = +\infty \). Clearly, since (locally) \( E = \{ x \in \Omega : v(x) - w(x) = -\infty \} \), the exceptional set \( E \) is a \( G_\delta \) of capacity zero (see [5, Section 5.9]). It is well-known that the distributional Laplacian \( \Delta u \) of \( u \) as above is a real Radon measure in \( \Omega \) (see [6, Theorem 4.1.8]). Conversely, if \( u \in \mathcal{D}'(\Omega) \) is such that \( \Delta u \) is a real Radon measure in \( \Omega \), then \( u \) can be written as a sum of a subharmonic and a superharmonic function in \( \Omega \) (see the proof of Theorem 2 below).

We recall that a function \( f : \mathbb{R} \to \mathbb{R} \) is said to be convex if the inequality \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \) holds for all \( x, y \in \mathbb{R} \) and \( 0 < t < 1 \). The (distributional) derivative \( f' \) of a convex function \( f \) is increasing and we set
\[
f'(x-) = \lim_{y \to x^-} f'(y) \quad \text{and} \quad f'(x+) = \lim_{y \to x^+} f'(y)
\]
for \( x \in \mathbb{R} \) and similarly
\[
f'(-\infty) = \lim_{x \to -\infty} f'(x) \quad \text{and} \quad f'(+\infty) = \lim_{x \to +\infty} f'(x).
\]
Below we shall consider convex functions \( f : \mathbb{R} \to \mathbb{R} \) that are of linear growth, that is, \( \sup_{x \in \mathbb{R}} |f(x)|/(1 + |x|) < \infty \). Note that for such an \( f \) the derivative \( f' \) is bounded. Indeed, for \( x > 0 \) we have that \( x f'(x) \leq \int_{x}^{2x} f'(t)dt = f(2x) - f(x) \) and similarly for \( x < 0 \).

Let \( \mu \) be a real Radon measure in \( \Omega \) and denote by \( |\mu| \) its total variation. The measures \( \mu^+ = (|\mu|+\mu)/2 \) and \( \mu^- = (|\mu|-\mu)/2 \) are called the positive and negative variations of \( \mu \), respectively, and the decomposition \( \mu = \mu^+ - \mu^- \) is commonly referred to as the Jordan decomposition of \( \mu \). The minimum property of the Jordan decomposition asserts that if \( \mu = \mu_1 - \mu_2 \), where \( \mu_j \geq 0 \) (\( j = 1, 2 \)), then \( \mu_1 \geq \mu^+ \) and \( \mu_2 \geq \mu^- \). A pair \((\Omega^+, \Omega^-)\) of measurable sets such that \( \Omega = \Omega^+ \cup \Omega^- \) and \( \Omega^+ \cap \Omega^- = \emptyset \) with the property that \( \mu^+ = \mu|\Omega^+ \) and \( \mu^- = -\mu|\Omega^- \) is called a Hahn decomposition of \( \Omega \) with respect to \( \mu \). Here \( \mu|E = \chi_E \mu \), where \( \chi_E \) denotes the characteristic function for the set \( E \). It is well-known that a Hahn decomposition exists and is uniquely determined up to sets of \( |\mu| \)-measure zero. We refer to [10, Chapter 6] for details.

We can now state the main result of this paper.

**Theorem 1.** Let \( u \) be a function in \( \Omega \) which is locally a sum of a subharmonic and a superharmonic function and denote by \( E \) an exceptional set for \( u \). Let
$f : \mathbb{R} \to \mathbb{R}$ be a convex function of linear growth. Then
\begin{equation}
\Delta(f \circ u) \geq \left( f'(-\infty)\chi_{\Omega^+} + f'(\infty)\chi_{\Omega^-} + f'(u+)\chi_{\Omega^+ \setminus E} + f'(u-)\chi_{\Omega^- \setminus E} \right) \Delta u \quad \text{in } D'(\Omega),
\end{equation}
where $(\Omega^+, \Omega^-)$ is a Hahn decomposition of $\Omega$ with respect to $\Delta u$.

**Remark 1.1.** Part of the conclusion of Theorem 1 is that the Laplacian $\Delta(f \circ u)$ is a Radon measure in $\Omega$. Indeed, inequality (1.1) means that
\begin{equation}
\Delta f(u) = \left( f'(-\infty)\chi_{\Omega^+} + f'(\infty)\chi_{\Omega^-} + f'(u+)\chi_{\Omega^+ \setminus E} + f'(u-)\chi_{\Omega^- \setminus E} \right) \Delta u + \mu,
\end{equation}
where $\mu$ is a positive Radon measure in $\Omega$.

The proof of Theorem 1 will be given in Section 2. In applying Theorem 1 it is of interest to have the set $E$ small. The smallest possible choice of exceptional set $E$ is given by (1.5) below.

A Radon measure $\mu$ in $\Omega$ is said to be **absolutely continuous** with respect to capacity (in symbols $\mu \ll \text{cap}$) if $|\mu|(E) = 0$ for every Borel set $E \subset \Omega$ such that $\text{cap}(E) = 0$. Similarly, a Radon measure $\mu$ in $\Omega$ is said to be **singular** with respect to capacity (in symbols $\mu \perp \text{cap}$) if $\mu$ is carried by a Borel subset of $\Omega$ of capacity zero, that is, $\mu = \mu|_{E}$, where $\text{cap}(E) = 0$. It is a known fact that every Radon measure $\mu$ in $\Omega$ has a **Lebesgue decomposition** with respect to capacity, that is, there exist Radon measures $\mu_a$ and $\mu_s$ in $\Omega$ such that
\begin{equation}
\mu = \mu_a + \mu_s, \quad \mu_a \ll \text{cap} \quad \text{and} \quad \mu_s \perp \text{cap}.
\end{equation}
For a proof we refer to [4, Lemma 2.1]. It is straightforward to see that for Radon measures $\mu$ and $\nu$ in $\Omega$ we have $\mu \leq \nu$ if and only if $\mu_a \leq \nu_a$ and $\mu_s \leq \nu_s$.

Passing to the absolutely continuous with respect to capacity parts in (1.1) we arrive at the following corollary.

**Corollary 1.** Let $u$ and $f$ be as in Theorem 1. Then
\begin{equation}
(\Delta(f \circ u))_a \geq \left( f'(u+)\chi_{\Omega^+} + f'(u-)\chi_{\Omega^-} \right) (\Delta u)_a \quad \text{in } D'(\Omega),
\end{equation}
where $(\Omega^+, \Omega^-)$ is a Hahn decomposition of $\Omega$ with respect to $\Delta u$.

**Proof.** The exceptional set $E$ in Theorem 1 has capacity zero. \(\square\)

Note that since $u$ is pointwise defined outside a set of capacity zero the right hand side in (1.2) is well-defined (cf. Corollary 2 below). A corresponding result for the singular with respect to capacity part $(\Delta(f \circ u))_s$ is given in Theorem 4 below.

The next corollary specializes to the case when the Riesz measure $\Delta u$ is absolutely continuous with respect to capacity.
Corollary 2. Let \( u \) and \( f \) be as in Theorem 1, and assume that \( \Delta u \) is absolutely continuous with respect to capacity. Then
\[
\Delta(f \circ u) \geq (f'(u^+) \chi_{\Omega^+} + f'(u^-) \chi_{\Omega^-}) \Delta u \quad \text{in } D'(\Omega),
\]
where \((\Omega^+, \Omega^-)\) is a Hahn decomposition of \( \Omega \) with respect to \( \Delta u \).

Proof. The exceptional set \( E \) in Theorem 1 has capacity zero. \( \square \)

We note also that Theorem 1 gives the following lower bound for \( \Delta(f \circ u) \) (cf. Theorem 4 below).

Corollary 3. Let \( u \) and \( f \) be as in Theorem 1. Then
\[
\Delta(f \circ u) \geq (f'(-\infty) \chi_{\Omega^+} + f'(\infty) \chi_{\Omega^-}) \Delta u \quad \text{in } D'(\Omega),
\]
where \((\Omega^+, \Omega^-)\) is a Hahn decomposition of \( \Omega \) with respect to \( \Delta u \).

Proof. The function \( f' \) is increasing. \( \square \)

Using Theorem 1 we can show that convex functions of linear growth operate on the class of \( \delta \)-subharmonic functions.

Theorem 2. Let \( u \) be a function in \( \Omega \) which is locally a sum of a subharmonic and a superharmonic function, and let \( f: \mathbb{R} \to \mathbb{R} \) be a convex function of linear growth. Then the composite function \( f \circ u \) is a sum of a subharmonic and a superharmonic function in \( \Omega \).

Proof. By Theorem 1 the distributional Laplacian \( \Delta(f \circ u) \) is a real Radon measure in \( \Omega \) (see Remark 1.1). By the Jordan decomposition (see [10, Chapter 6]) there exist positive Radon measures \( \mu_j \) \((j = 1, 2)\) in \( \Omega \) such that \( \Delta(f \circ u) = \mu_1 - \mu_2 \) in \( D'(\Omega) \). By [6, Theorem 4.4.6] we can find \( u_j \in D'(\Omega) \) such that \( \Delta u_j = \mu_j \) in \( D'(\Omega) \) \((j = 1, 2)\). The distribution \( u_j \) is naturally identified with a subharmonic function in \( \Omega \) (see [6, Theorem 4.1.8]) and by Weil’s lemma (see [6, Theorem 4.4.1]) we conclude that \( f \circ u = u_1 - u_2 + h \), where \( h \) is harmonic in \( \Omega \). \( \square \)

Recall that the fundamental solution \( \Gamma \) of the Laplacian is given by \( \Gamma(x) = \log |x|/(2\pi) \) for \( 0 \neq x \in \mathbb{R}^2 \) and
\[
\Gamma(x) = -|x|^{2-n}/((n-2)c_n) \quad \text{for } 0 \neq x \in \mathbb{R}^n, \ n \geq 3,
\]
where \( c_n = 2\pi^{n/2}/\Gamma(n/2) \) is the area of the unit sphere in \( \mathbb{R}^n \) (see [6, Theorem 3.3.2]). The F. Riesz representation formula for subharmonic functions asserts that every function \( u \) subharmonic in a neighborhood of a ball \( B \) has the form
\[
u(x) = \int_B \Gamma(x-y)\Delta u(dy) + h(x), \quad x \in B,
\]
where $h$ is harmonic in $B$ (see [5, Theorem 3.9]). The formula (1.4) is easy to prove using distribution theory (see [6, Section 4.4]).

Let $u \in \mathcal{D}'(\Omega)$ be such that $\Delta u = \mu$ is a real Radon measure in $\Omega$. As in the proof of Theorem 2 (here with $f(x) = x$ for $x \in \mathbb{R}$), proceeding from the Jordan decomposition of $\mu$, we can write $u$ globally as a sum $u = v + w$ of a subharmonic function $v$ in $\Omega$ and a superharmonic function $w$ in $\Omega$ in such a way that $\Delta v = \mu^+$ and $\Delta w = -\mu^-$ in $\mathcal{D}'(\Omega)$. By the F. Riesz formula (1.4) the exceptional set $E$ for $u$ relative to this sum decomposition is given by

\begin{equation}
E = \left\{ x \in \Omega : \int_{B(x,r)} \Gamma(x-y)d|\mu|(y) = -\infty \text{ for } 0 < r < \text{dist}(x, \partial\Omega) \right\},
\end{equation}

where $\Gamma$ is the fundamental solution of the Laplacian.

Conversely, if $u = v + w$, where $v$ is subharmonic and $w$ is superharmonic in $\Omega$, then $\Delta u = \mu = \Delta v + \Delta w$ in $\mathcal{D}'(\Omega)$ and the minimum property of the Jordan decomposition yields that $\Delta v \geq \mu^+$ and $-\Delta w \geq \mu^-$ in $\mathcal{D}'(\Omega)$. By the F. Riesz formula (1.4), these inequalities make evident that the choice of exceptional set $E$ in (1.5) for $u$ is the smallest possible.

The next theorem specializes to subharmonic functions.

**Theorem 3.** Let $u$ be subharmonic in $\Omega$, and let $f : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. Then

\begin{equation}
\Delta(f \circ u) \geq f'(u^+)\Delta u \text{ in } \mathcal{D}'(\Omega).
\end{equation}

**Proof.** For $\lambda > 0$ we consider the function $f_\lambda$ defined by $f_\lambda(x) = f(x)$ for $x < \lambda$ and $f_\lambda(x) = f'(\lambda^-)(x-\lambda) + f(\lambda)$ for $x \geq \lambda$. Clearly, $f_\lambda$ is convex and of linear growth. By Theorem 1 we have that

\begin{equation}
\Delta(f_\lambda \circ u) \geq f_\lambda'(u^+)\Delta u \text{ in } \mathcal{D}'(\Omega).
\end{equation}

Let now $G$ be an open set with compact closure contained in $\Omega$. By upper semicontinuity $u$ is bounded from above in $G$. Now, clearly, $f_\lambda \circ u = f \circ u$ in $G$ for all $\lambda > 0$ sufficiently large, and we have that $\Delta(f_\lambda \circ u) = \Delta(f \circ u)$ in $\mathcal{D}'(G)$ for such $\lambda$. A straightforward computation (see [6, Theorem 3.1.3]) shows that the distributional derivative $f_\lambda'$ is given by

\[
f_\lambda'(x) = \begin{cases} 
f'(x) & \text{for } x < \lambda \\
f'(\lambda^-) & \text{for } x > \lambda. \end{cases}
\]

As above, by upper semicontinuity, we have that $f_\lambda'(u(x)^+) = f'(u(x)^+)$ for $x \in G$ provided $\lambda > 0$ is sufficiently large. Inequality (1.7) now yields that $\Delta(f \circ u) \geq f'(u^+)\Delta u$ in $\mathcal{D}'(G)$. Since $G$ is arbitrary (1.6) follows. \qed
We remark that we have written \( f'(-\infty^+) = f'(-\infty) \) in (1.6).

It is well-known that \( f \circ u \) is subharmonic if \( u \) is subharmonic and \( f \) is convex and increasing (see [5, Theorem 2.2]). It is also well-known that subharmonic functions naturally corresponds to distributions with nonnegative Laplacian (see [6, Theorem 4.1.8]). In this context inequality (1.6) gives an improved lower bound of the Riesz mass \( \Delta(f \circ u) \) for the subharmonic function \( f \circ u \).

We now consider the singular with respect to capacity part of \( \Delta(f \circ u) \).

**Theorem 4.** Let \( u \) and \( f \) be as in Theorem 1. Then
\[
(\Delta(f \circ u))_s = (f'(-\infty)\chi_{\Omega^+} + f'(\infty)\chi_{\Omega^-})(\Delta u)_s \text{ in } D'(\Omega),
\]
where \((\Omega^+, \Omega^-)\) is a Hahn decomposition of \( \Omega \) with respect to \( \Delta u \).

**Remark 1.2.** Recall Corollary 3. The principal conclusion of Theorem 4 is that when restricting to a set of capacity zero we get equality in (1.3).

The proof of Theorem 4 will be given in Section 2. For the choice \( f(x) = x^+ = \max(x, 0) \) for \( x \in \mathbb{R} \) the above theorem is due to Brezis and Ponce (see [1, Theorem 1.1 (6)]).

2. Estimation of Riesz mass

The main task of this section is to prove Theorem 1. For the proof of this result we need some preparation. We recall that a point \( x \in \Omega \) is a Lebesgue point for \( u \in L^1_{\text{loc}}(\Omega) \) if
\[
\lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |u(y) - u(x)| dy = 0,
\]
where we have used \( \int \) to denote integration with respect to normalized measure. The next lemma is well-known. For the sake of completeness we include some details of proof.

**Lemma 2.1.** Let \( u \) be as in Theorem 1. Then every point \( x \in \Omega \) not in the exceptional set \( E \) of \( u \) is a Lebesgue point for \( u \).

**Proof.** Clearly we can assume that \( u \) is subharmonic. Let \( u^+ = \max(u, 0) \) and \( u^- = -\min(u, 0) \), so that \( u = u^+ - u^- \) and \( u^+, u^- \geq 0 \). Note that
\[
\int_{B(x, r)} |u(y) - u(x)| dy = \int_{B(x, r)} (u(y) - u(x))^+ dy + \int_{B(x, r)} (u(y) - u(x))^- dy.
\]
By upper semicontinuity we have that \( \lim_{r \to 0} \int_{B(x, r)} (u(y) - u(x))^+ dy = 0 \). Also, since \( \int_{B(x, r)} (u(y) - u(x))^- dy = \int_{B(x, r)} (u(y) - u(x))^+ dy - \int_{B(x, r)} (u(y) - u(x))^+ dy \), the submean inequality yields that \( \lim_{r \to 0} \int_{B(x, r)} (u(y) - u(x))^+ dy = 0 \). \( \square \)
In the proof of Theorem 1 we will use certain regularizations of \( u \) that we now proceed to define. Let \( \psi \in \mathcal{D}(B(0, 1)) \) be a nonnegative test function such that \( \int \psi(x) dx = 1 \), and set \( \psi(x) = \psi(x/\varepsilon)/\varepsilon^n \) for \( \varepsilon > 0 \). For \( u \in L^1_{\text{loc}}(\Omega) \) we define \( u_\varepsilon \) by

\[
(2.1) \quad u_\varepsilon(x) = u * \psi_\varepsilon(x) = \int u(x - y) \psi_\varepsilon(y) dy \quad \text{for } x \in \Omega_\varepsilon,
\]

where \( \Omega_\varepsilon = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon \} \). Clearly \( u_\varepsilon \) is of type \( C^\infty \). Standard facts about these regularizations are that \( u_\varepsilon \to u \) in \( L^1_{\text{loc}}(\Omega) \) and \( u_\varepsilon(x) \to u(x) \) at every Lebesgue point \( x \) for \( u \).

**Lemma 2.2.** Let \( f \) be a continuous function on \( \mathbb{R} \) and let \( u \in L^1_{\text{loc}}(\Omega) \) be real-valued. Then at every Lebesgue point \( x \) for \( u \) we have that

\[
\lim_{\varepsilon \to 0} \int f(u_\varepsilon(y)) \psi_\varepsilon(y - x) dy = f(u(x)),
\]

where \( u_\varepsilon \) is as in (2.1).

**Proof.** Note that \( u_\varepsilon(y) - u(x) = \int (u(y - t) - u(x)) \psi_\varepsilon(t) dt \). For \( y \) with \( |y - x| < \varepsilon \) we have that

\[
|u_\varepsilon(y) - u(x)| \leq C \varepsilon^n \int_{B(y, \varepsilon)} |u(t) - u(x)| dt \leq C \varepsilon^n \int_{B(x, 2\varepsilon)} |u(t) - u(x)| dt \to 0.
\]

Continuity of \( f \) at \( u(x) \) now yields that \( \int f(u_\varepsilon(y)) \psi_\varepsilon(y - x) dy = f(u(x)) + o(1) \).

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** The proof will proceed in several steps. First let \( f \) and \( u \) be of type \( C^2 \). A computation shows that \( \Delta(f \circ u) = f''(u)|\nabla u|^2 + f'(u)\Delta u \). By this equality we see that \( \Delta(f \circ u) \geq f'(u)\Delta u \) when both \( f \) and \( u \) are of type \( C^2 \).

Next we consider the case when \( f \) is of type \( C^2 \) and \( u \) is as in the theorem. We prove that then

\[
(2.2) \quad \Delta(f \circ u) \geq \left( f'(\infty)\chi_{E \cap \Omega^+} + f'(\infty)\chi_{E \cap \Omega^-} + f'(u)\chi_{\Omega \setminus E} \right) \Delta u \quad \text{in } \mathcal{D}'(\Omega).
\]

Let \( u_\varepsilon \) be as in (2.1). By the special case of (1.1) already proved we know that

\[
(2.3) \quad \Delta(f \circ u_\varepsilon) \geq f'(u_\varepsilon)\Delta u_\varepsilon \quad \text{in } \mathcal{D}'(\Omega_\varepsilon).
\]

Since \( f' \) is bounded and \( u_\varepsilon \to u \) in \( L^1_{\text{loc}}(\Omega) \) we have \( f(u_\varepsilon) \to f(u) \) in \( L^1_{\text{loc}}(\Omega) \), which yields that \( \Delta(f \circ u_\varepsilon) \to \Delta(f \circ u) \) in \( \mathcal{D}'(\Omega) \). We now turn our attention to
the term \( f'(u_\varepsilon) \Delta u_\varepsilon \) in (2.3). Let \( 0 \leq \varphi \in C_c(\Omega) \) and set \( \mu = \Delta u \). By Fubini’s theorem we have that

(2.4) \[
\langle f'(u_\varepsilon) \Delta u_\varepsilon, \varphi \rangle = \int I_\varepsilon(x) d\mu(x) = \left( \int_{E \cap \Omega^+} + \int_{E \cap \Omega^-} + \int_{\Omega \setminus E} \right) I_\varepsilon(x) d\mu(x),
\]

where \( I_\varepsilon(x) = \int \varphi(y) f'(u_\varepsilon(y)) \psi_\varepsilon(y-x) dy \).

We next estimate the integral \( \int_{E \cap \Omega^+} I_\varepsilon d\mu \). We note that the inequality \( I_\varepsilon(x) \geq f'(-\infty) \int \varphi(y) \psi_\varepsilon(y-x) dy \) yields that

\[
\int_{E \cap \Omega^+} I_\varepsilon(x) d\mu(x) \geq f'(-\infty) \int_{E \cap \Omega^+} (\varphi \ast \hat{\psi}_\varepsilon)(x) d\mu(x), \]

where we have written \( \hat{\psi}_\varepsilon(x) = \psi(-x) \). Similarly we have that \( \int_{E \cap \Omega^-} I_\varepsilon d\mu \geq f'(\infty) \int_{E \cap \Omega^-} \varphi \ast \hat{\psi}_\varepsilon d\mu \).

Next we consider the integral \( \int_{\Omega \setminus E} I_\varepsilon d\mu \). For \( \varepsilon > 0 \) small \( I_\varepsilon \) has support contained in a fixed compact subset of \( \Omega \) and is bounded there uniformly in \( \varepsilon > 0 \). Note that \( I_\varepsilon(x) = o(1) + \varphi(x) \int f'(u_\varepsilon(y)) \psi_\varepsilon(y-x) dy \). Now Lemma 2.1 and Lemma 2.2 yield that \( I_\varepsilon(x) \to \varphi(x)f'(u(x)) \) for \( x \in \Omega \setminus E \). By the Lebesgue dominated convergence theorem we conclude that \( \int_{\Omega \setminus E} I_\varepsilon d\mu \to \int_{\Omega \setminus E} \varphi f(u) d\mu \).

We now finish the proof of (2.2). By (2.3), (2.4) and the above estimates of \( \int_{E \cap \Omega^+} I_\varepsilon d\mu \) and \( \int_{E \cap \Omega^-} I_\varepsilon d\mu \) we have that

\[
\langle \Delta(f \circ u_\varepsilon), \varphi \rangle \geq f'(-\infty) \int_{E \cap \Omega^+} \varphi \ast \hat{\psi}_\varepsilon d\mu + f'(\infty) \int_{E \cap \Omega^-} \varphi \ast \hat{\psi}_\varepsilon d\mu + \int_{\Omega \setminus E} I_\varepsilon d\mu.
\]

Now using the result of the previous paragraph that \( \int_{\Omega \setminus E} I_\varepsilon d\mu \to \int_{\Omega \setminus E} \varphi f(u) d\mu \) together with the standard fact that \( \varphi \ast \hat{\psi}_\varepsilon \to \varphi \) in \( C_c(\Omega) \) a passage to the limit as \( \varepsilon \to 0 \) yield (2.2).

Next we consider the general case when \( f \) and \( u \) both are as in the theorem. Let \( 0 \leq \psi_1 \in \mathcal{D}(0,1) \) be such that \( \int \psi_1(x) dx = 1 \) and consider the regularizations \( f_\varepsilon \) defined by \( f_\varepsilon(x) = \int f(x - \varepsilon y) \psi_1(y) dy \) for \( x \in \mathbb{R} \). The function \( f_\varepsilon \) is smooth, convex and of linear growth. Since \( f_\varepsilon(x) - f(x) = \int (f(x - \varepsilon y) - f(x)) \psi_1(y) dy \) and \( f' \) is bounded, \( f_\varepsilon \) converges to \( f \) uniformly on \( \mathbb{R} \). We also note that \( f'_\varepsilon(-\infty) = f'(-\infty) \) and \( f'_\varepsilon(\infty) = f'(\infty) \). By (2.2) we have that

(2.5) \[
\Delta(f_\varepsilon \circ u) \geq (f'(-\infty) \chi_{E \cap \Omega^+} + f'(\infty) \chi_{E \cap \Omega^-}) \Delta u + f'_\varepsilon(u) \chi_{\Omega \setminus E} \Delta u \quad \text{in } \mathcal{D}'(\Omega).
\]

By uniform convergence it is clear that \( f_\varepsilon \circ u \to f \circ u \) in \( L^1_{\text{loc}}(\Omega) \), which yields that \( \Delta(f_\varepsilon \circ u) \to \Delta(f \circ u) \) in \( \mathcal{D}'(\Omega) \).
We now consider the term $f'_x(u)\chi_{\Omega \setminus E} \Delta u$ in (2.5). Note that $f'_x(x) = \int f'(x - \varepsilon y) \psi_1(y)dy$ for $x \in \mathbb{R}$. Since $f'$ is increasing, $f'_x(x)$ increases to $f'(x-)$ as $\varepsilon \to 0+$, and by Lebesgue's theorem we have that $f'_x(u)\chi_{\Omega \setminus E} \Delta u \to f'(u-)\chi_{\Omega \setminus E} \Delta u$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0+$. Similarly we have that $f'_x(u)\chi_{\Omega \setminus E} \Delta u \to f'(u+)\chi_{\Omega \setminus E} \Delta u$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0-$. 

We now finish the proof of (1.1). Letting $\varepsilon \to 0+$ in (2.5) we obtain that

\[
\Delta(f \circ u) \geq (f'(-\infty)\chi_{E \cap \Omega^+} + f'(\infty)\chi_{E \cap \Omega^-}) \Delta u + f'(u-)\chi_{\Omega \setminus E} \Delta u \quad \text{in } \mathcal{D}'(\Omega).
\]

Multiplying this inequality by $\chi_{\Omega^-}$ we obtain that

\[
\chi_{\Omega^-} \Delta(f \circ u) \geq f'(\infty)\chi_{E \cap \Omega^-} \Delta u + f'(u-)\chi_{\Omega^- \setminus E} \Delta u \quad \text{in } \mathcal{D}'(\Omega).
\]

Similarly, by letting $\varepsilon \to 0-$ in (2.5) and multiplying by $\chi_{\Omega^+}$, we have that

\[
\chi_{\Omega^+} \Delta(f \circ u) \geq f'(-\infty)\chi_{E \cap \Omega^+} \Delta u + f'(u+)\chi_{\Omega^+ \setminus E} \Delta u \quad \text{in } \mathcal{D}'(\Omega).
\]

Now by adding (2.6) and (2.7) we obtain (1.1). \hfill \Box

We now turn to the proof of Theorem 4. We shall use the following lemma which has recently appeared in a paper by Dupaigne and Ponce (see [3, Theorem 3]). For the sake of completeness we include some details of proof.

**Lemma 2.3.** Let $u$ be a sum of a subharmonic and a superharmonic function in $\Omega$ such that $u(x) \geq 0$ for $x \in \Omega \setminus E$, where $E$ is an exceptional set for $u$. Then $(\Delta u)|_F \leq 0$ in $\mathcal{D}'(\Omega)$ for every Borel set $F \subset \Omega$ such that $\text{cap}(F) = 0$.

**Proof.** Let $K$ be a compact subset of $\Omega$ of capacity zero, and let $\Delta u = \mu$. It suffices to show that $\mu^+|_K = 0$. By the assumption $u \geq 0$ in $\Omega \setminus E$ it is straightforward to see that using the F. Riesz formula (1.4) that the measure $\mu^+|_K$ has finite energy:

\[
\int_K \int_K \Gamma(x-y)d\mu^+(x)d\mu^+(y) > -\infty,
\]

where $\Gamma$ is the fundamental solution of the Laplacian. Since $K$ has capacity zero this implies that $\mu^+|_K = 0$ (see [5, Section 5.1]). \hfill \Box

We can now prove Theorem 4.

**Proof of Theorem 4.** Let $E \subset \Omega$ be a Borel set of capacity zero. By Corollary 3 it suffices to prove the inequality

\[
\Delta(f \circ u)|_E \leq (f'(-\infty)\chi_{E^+} + f'(\infty)\chi_{E^-}) \Delta u|_E \quad \text{in } \mathcal{D}'(\Omega)
\]

(see Remark 1.2). Let $u_0 \in \mathbb{R}$. By convexity of $f$ we know that

\[
f \circ u(x) \geq f'(u_0+)u(x) - u_0) + f(u_0) \quad \text{for } x \in \Omega,
\]
and an application of Lemma 2.3 gives that $Δ(f \circ u)|E \leq f'(u_0+)Δu|E$ in $D'(Ω)$.
Letting $u_0 \to +\infty$ we see that
\begin{equation}
Δ(f \circ u)|E \leq f'(+∞)Δu|E \quad \text{in } D'(Ω).
\end{equation}
Similarly, letting $u_0 \to -\infty$ we see that
\begin{equation}
Δ(f \circ u)|E \leq f'(-∞)Δu|E \quad \text{in } D'(Ω).
\end{equation}
By (2.9) and (2.10) we now conclude that
\begin{equation*}
Δ(f \circ u)|E = χ_{Ω^+}Δ(f \circ u)|E + χ_{Ω^-}Δ(f \circ u)|E \leq f'(-∞)χ_{Ω^+}Δu|E + f'(+∞)χ_{Ω^-}Δu|E \quad \text{in } D'(Ω),
\end{equation*}
which proves (2.8). \qed

3. A comparison with earlier results

The case when $f(x) = x^+ = \max(x, 0)$ for $x \in ℝ$ has recently been studied by Brezis and Ponce [1]. In this context (1.2) reduces to the inequality
\begin{equation}
(Δu^+)_a = (χ_{\{u \geq 0\}} Δu^+ + χ_{\{u > 0\}} Δu^-)(Δu)_a \quad \text{in } D'(Ω),
\end{equation}
where $(Ω^+, Ω^-)$ is a Hahn decomposition of $Ω$ with respect to $Δu$. Note that
\begin{equation*}
χ_{\{u \geq 0\}} Δu^+ + χ_{\{u > 0\}} Δu^- = χ_{\{u = 0\}} Δu^+ + χ_{\{u > 0\}} = -χ_{\{u = 0\}} Δu^- + χ_{\{u \geq 0\}}.
\end{equation*}
It is thus clear that (3.1) implies the inequalities
\begin{equation}
(Δu^+)_a \geq χ_{\{u \geq 0\}} (Δu)_a \quad \text{and } (Δu^-)_a \geq χ_{\{u > 0\}} (Δu)_a \quad \text{in } D'(Ω),
\end{equation}
which were established in [1, Theorem 1.1 (5), Remark 1 (9)]. Conversely, the inequalities (3.2) imply (3.1). Indeed, by (3.2) we have
\begin{equation*}
(Δu^+)_a = χ_{Ω^+} (Δu^+)_a + χ_{Ω^-} (Δu^-)_a \geq χ_{\{u \geq 0\}} Δu^+ + χ_{\{u > 0\}} Δu^- \quad \text{in } D'(Ω),
\end{equation*}
which yields (3.1).

We now consider the case when $f(x) = |x|$ for $x \in ℝ$ and $Δu$ is absolutely continuous with respect to capacity. Corollary 2 now reduces to
\begin{equation}
 Δ|u| \geq (χ_{\{u = 0\}} Δu^+ - χ_{\{u = 0\}} Δu^- + \text{sgn}(u)) Δu \quad \text{in } D'(Ω),
\end{equation}
where \( \text{sgn}(x) = x/|x| \) for $0 \neq x \in ℝ$ and \( \text{sgn}(0) = 0 \). Clearly (3.3) implies that
\begin{equation}
 Δ|u| \geq \text{sgn}(u) Δu \quad \text{in } D'(Ω).
\end{equation}
We mention here that specialized to our context the result by Kato [7, Lemma A] asserts that (3.4) holds when $Δu$ is absolutely continuous with respect to Lebesgue measure in $Ω$. 

We now give an example where we have equality in (3.3) and not equality in (3.4). Let $u(z) = \log^+ |z|$ for $z \in \mathbb{C}$. It is then straightforward to see that $\Delta u$ is the arc length measure on the unit circle. The details are omitted.

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