A monotonicity estimate of the biharmonic Green function

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Abstract. In this note we present a refined form of the monotonicity property for the biharmonic Green function and the related so called harmonic compensator. Specialized to the harmonic compensator the estimate obtained is shown to be sharp.

0. Introduction. Let \( \mathbb{D} \) be the unit disc in the complex plane and denote by \( \mathbb{T} = \partial \mathbb{D} \) the unit circle. The biharmonic Green function is the function \( \Gamma \) defined by

\[
\Gamma(z, \zeta) = |z - \zeta|^2 \log \left| \frac{z - \zeta}{1 - z\zeta} \right|^2 + (1 - |z|^2)(1 - |\zeta|^2), \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.
\]

The related function \( H \) defined by

\[
H(z, e^{i\theta}) = \frac{(1 - |z|^2)^2}{|e^{i\theta} - z|^2}, \quad (z, e^{i\theta}) \in \mathbb{D} \times \mathbb{T},
\]

is known as the harmonic compensator. Recently, these two functions have attracted special attention because of their relevance to Bergman space theory (see [5]).

The principal new result in this note is a sharp form of the monotonicity property for the harmonic compensator \( H \) (see Theorem 1.1 and Remark 1.1). By known arguments this estimate for the harmonic compensator leads to a corresponding estimate for the biharmonic Green function and more generally to an estimate for functions from a certain class of nonnegative super-biharmonic functions (see Corollary 1.1 and Corollary 1.2). The monotonicity property for the biharmonic Green function originates from the work [3] by Aleman, Richter and Sundberg. An important point is that estimates of this type yield norm estimates for dilation operators arising in approximation problems (see for instance Section 2).

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The setup is as follows. We write \( dA = dx dy / \pi \) for the planar Lebesgue measure normalized so that \( \mathbb{D} \) has unit area. By \( \Delta \) we denote the normalized laplacian defined by
\[
\Delta = \partial^2 / \partial z \partial \bar{z},
\]
where
\[
\partial / \partial z = (\partial / \partial x - i \partial / \partial y) / 2 \quad \text{and} \quad \partial / \partial \bar{z} = (\partial / \partial x + i \partial / \partial y) / 2
\]
are the usual complex derivatives. The real and imaginary part of a complex number \( z \) are denoted by \( \Re(z) \) and \( \Im(z) \), respectively.

1. Monotonicity estimates. We turn to estimates of \( H \). First we need a lemma.

**Lemma 1.1.** The harmonic compensator \( H \) admits the estimate

\[
\Re \left( \left( \frac{\partial H}{\partial z}(z, e^{i\theta}) \right) \right) \leq \frac{|z|}{1 + |z|} H(z, e^{i\theta}), \quad (z, e^{i\theta}) \in \mathbb{D} \times \mathbb{T}.
\]

**Proof.** We write \( P(z, e^{i\theta}) = (1 - |z|^2)/|e^{i\theta} - z|^2 \) for the usual Poisson kernel for the unit disc. Note that \( H(z, e^{i\theta}) = (1 - |z|^2) P(z, e^{i\theta}) \). A computation shows that

\[
\left( \frac{\partial H}{\partial z}(z, e^{i\theta}) \right) = -|z|^2 P(z, e^{i\theta}) + (1 - |z|^2) \frac{e^{i\theta} z}{|e^{i\theta} - z|^2}.
\]

We now have that

\[
\Re \left( \left( \frac{\partial H}{\partial z}(z, e^{i\theta}) \right) \right) \leq -|z|^2 P(z, e^{i\theta}) + (1 - |z|^2) \frac{|z|}{|e^{i\theta} - z|^2} = \frac{|z|}{1 + |z|} H(z, e^{i\theta}),
\]

which yields the conclusion of the lemma. \( \square \)

We have the following theorem.

**Theorem 1.1.** The harmonic compensator \( H \) admits the estimate

\[
\left( \frac{r}{r + |z|} \right)^2 H(z/r, e^{i\theta}) \leq \frac{1}{(1 + |z|)^2} H(z, e^{i\theta}), \quad |z| < r < 1, \quad e^{i\theta} \in \mathbb{T}.
\]

In particular, we have that

\[
H(z/r, e^{i\theta}) \leq \frac{4}{(1 + r)^2} H(z, e^{i\theta}), \quad |z| < r < 1, \quad e^{i\theta} \in \mathbb{T}.
\]

**Proof.** We first compute that

\[
\frac{d}{dr} H(z/r, e^{i\theta}) = \frac{z}{r^2} \frac{\partial H}{\partial z}(z/r, e^{i\theta}) - \frac{\bar{z}}{r^2} \frac{\partial H}{\partial \bar{z}}(z/r, e^{i\theta})
\]

\[
= -\frac{2}{r} \Re \left( \left( \frac{\partial H}{\partial z}(z/r, e^{i\theta}) \right) \right).
\]
An application of Lemma 1.1 yields the inequality
\[ \frac{d}{dr} H(z/r, e^{i\theta}) \geq -2 \frac{|z|}{r(r + |z|)} H(z/r, e^{i\theta}). \]

This last differential inequality can easily be reformulated as
\[ (1.3) \quad \frac{d}{dr} \left[ \left( \frac{r}{r + |z|} \right)^2 H(z/r, e^{i\theta}) \right] \geq 0. \]

Clearly inequality (1.3) implies (1.1).

To prove (1.2) we note that
\[ \frac{|z|}{r + |z|} \leq \frac{2r}{1 + r} \quad \text{for} \quad |z| < r < 1. \]

Thus (1.1) implies (1.2).

Remark 1.1. We note that (1.2) gives the best possible estimate of the form
\[ H(z/r, e^{i\theta}) \leq c(r) H(z, e^{i\theta}), \quad |z| < r < 1, \quad e^{i\theta} \in \mathbb{T}. \]

(1.4) Indeed, we have that
\[ \frac{1}{r} = \sum_{k \geq 0} (1 - r)^k \quad \text{and} \quad \frac{4}{(1 + r)^2} = \sum_{k \geq 0} \frac{k + 1}{2^k} (1 - r)^k \]

differ from the second term and on.

We record two corollaries of Theorem 1.1.

**Corollary 1.1.** The biharmonic Green function \( \Gamma \) admits the estimate
\[ \Gamma(z/r, \zeta) \leq \frac{4}{(1 + r)^2} \Gamma(z, \zeta), \quad |z| < r < 1, \quad \zeta \in \mathbb{D}. \]

**Sketch of proof.** The biharmonic Green function can be expressed in terms of the harmonic compensator by the formula
\[ \Gamma(z, \zeta) = \frac{1}{\pi} \int_{\max(|z|,|\zeta|)}^{1} \int_{\mathbb{T}} H(z/s, e^{i\theta}) H(\zeta/s, e^{i\theta}) d\theta ds, \quad (z, \zeta) \in \mathbb{D}^2 \]

(see [5, page 64], the formula originates from [4]). Using this formula it is easy to see that (1.2) yields the conclusion of the corollary (see [1, proof of Lemma 2.3] or [5, proof of Lemma 3.14]).

A distribution \( u \) is said to be super-biharmonic in \( \mathbb{D} \) if the (distributional) bilaplacian \( \Delta^2 u \)
is a positive Radon measure in \( \mathbb{D} \). By regularity theory, a super-biharmonic distribution can be defined by a function with interior regularity \( C^{1,\alpha} \) for every \( 0 < \alpha < 1 \).
Corollary 1.2. Let \( u \) be a nonnegative super-biharmonic function in \( D \) such that \( \lim_{r \to 1^-} \frac{1}{2\pi} \int_{\mathbb{T}} u(re^{i\theta}) d\theta = 0 \). Then \( u \) admits the estimate

\[
u(z/r) \leq \frac{4}{(1+r)^2} u(z), \quad |z| < r < 1.
\]

Sketch of proof. By [2, Corollary 3.7] the function \( u \) has the representation

\[
u(z) = \int_D \Gamma(z, \zeta) d\mu(\zeta) + \int_{\mathbb{T}} H(z, e^{i\theta}) d\lambda(e^{i\theta}), \quad z \in D,
\]

where \( \mu = \Delta^2 u \) and \( \lambda \) is a positive (regular) Borel measure on \( \mathbb{T} \). By this representation formula, inequality (1.2) and Corollary 1.1 make evident the conclusion of the corollary.

2. Dilation operators. The monotonicity estimates in Section 1 yield some better estimates of norms of dilation operators arising in the context of the Bergman space. As an example we consider a recent result by Abkar [1].

Let \( \omega : D \to (0, \infty) \) be a continuous weight function and \( 0 < p < \infty \). We denote by \( A^p(D, \omega) \) the weighted Bergman space of all \( p \)-th power integrable with respect to \( \omega \) analytic functions in \( D \), that is, a function \( f \) is in \( A^p(D, \omega) \) if and only if \( f \) is analytic in \( D \) and

\[
\| f \|_{p, \omega}^p = \int_{D} |f(z)|^p \omega(z) dA(z) < +\infty.
\]

We have the following proposition.

Proposition 2.1. Let \( 0 < p < \infty \). Assume that the weight function \( \omega : D \to (0, \infty) \) is super-biharmonic in \( D \) and such that \( \lim_{r \to 1^-} \frac{1}{2\pi} \int_{\mathbb{T}} u(re^{i\theta}) d\theta = 0 \). Then

\[
\| f_r \|_{p, \omega}^p \leq \frac{4}{r^2(1+r)^2} \| f \|_{p, \omega}^p, \quad f \in A^p(D, \omega),
\]

where \( f_r \) is defined by \( f_r(z) = f(rz) \) for \( z \in D \) and \( 0 < r < 1 \).

Proof. We have that

\[
\| f_r \|_{p, \omega}^p = \frac{1}{r^2} \int_{D} |f(z)|^p \omega(z/r) dA(z)
\]

\[
\leq \frac{4}{r^2(1+r)^2} \int_{D} |f(z)|^p \omega(z) dA(z) \leq \frac{4}{r^2(1+r)^2} \| f \|_{p, \omega}^p,
\]

where the first inequality follows by Corollary 1.2. \( \square \)
We remark that by standard arguments the norm estimate in Proposition 2.1 yields that the polynomials are dense in $A^p(D, \omega)$. We refer to [1] for details. The same arguments apply in the case of harmonic Bergman spaces.

References


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