The Dirichlet problem for standard weighted Laplacians in the upper half plane

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Abstract

In this paper the Dirichlet problem for a class of standard weighted Laplace operators in the upper half plane is solved by means of a counterpart of the classical Poisson integral formula. Boundary limits and representations of the associated solutions are studied within a framework of weighted spaces of distributions. Special attention is given to the development of a suitable uniqueness theory for the Dirichlet problem under appropriate growth constraints at infinity.

\textbf{Keywords:} Poisson integral, weighted Laplace operator, Poisson kernel, weighted space of distributions

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Introduction

Let $\Omega$ be a domain in the complex plane $\mathbb{C}$ and consider the weighted area element

$$dA_w(z) = w(z)dA(z), \quad z = x + iy \in \Omega,$$

where $w : \Omega \to (0, \infty)$ is a given weight function and $dA(z) = dx dy$ is the usual planar Lebesgue area element. Associated to $dA_w(z)$ is a weighted Laplace differential operator

$$\Delta_{w,z} = \partial_z w(z)\partial_z^{-1}, \quad z \in \Omega,$$

where $\partial$ and $\bar{\partial}$ are the usual complex derivatives. In fact, the kernel function of the norm

$$\int_\Omega |\phi(z)|dA_w(z)$$

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of holomorphic functions $\phi$ can be expressed in terms of the Green’s function of the differential equation $\Delta_{w,z} u(z) = 0$ in $\Omega$, see Garabedian [14] wherein the systematic study of weighted Laplacians of the form (0.1) seems to have been initiated.

Let $\text{Im} z$ denote the imaginary part of the complex number $z \in \mathbb{C}$. In the study of Bergman spaces in the upper half plane $\mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \}$ one often considers the so-called standard weight functions $w_\alpha(z) = (\text{Im} z)^\alpha$, where $\alpha$ is a real parameter. To simplify notation we shall let $\Delta_\alpha$ denote the weighted Laplace differential operator corresponding to the weight $w_\alpha$:

$$\Delta_{\alpha,z} = \bar{\partial}_z y^{-\alpha} \partial_z, \quad z = x + iy \in \mathbb{C}^+.$$ 

Note that for $\alpha \neq 0$, the operator $\Delta_\alpha$ has a certain degenerate or singular behavior on the boundary $\mathbb{R} = \partial \mathbb{C}^+$. The aim of the present paper is the study of the $\alpha$-harmonic equation

$$\Delta_\alpha u = 0 \quad \text{in} \quad \mathbb{C}^+, \quad (0.2)$$

solutions of which will be referred to as $\alpha$-harmonic functions. The main focus will be to solve the associated Dirichlet boundary value problem

$$\begin{cases} 
\Delta_\alpha u = 0 & \text{in} \quad \mathbb{C}^+, \\
u = f & \text{on} \quad \mathbb{R}. 
\end{cases} \quad (0.3)$$

Here the boundary condition is verified as a distributional limit, i.e. $\lim_{y \to 0} u_y = f$ in $\mathcal{S}'$, where

$$u_y(x) = u(x + iy), \quad x \in \mathbb{R}, \quad (0.4)$$

for $y > 0$, and $\mathcal{S}'$ is the space of tempered distributions on $\mathbb{R}$. We mention that the corresponding problem in the unit disc was solved in Olofsson and Wittsten [23]. For other related work in the unit disc setup, see Behm [6] where the Poisson equation is solved by means of Green’s function; see also Olofsson [22], as well as Borichev and Hedenmalm [8] and Hedenmalm [15], where the real part of this operator appears.

The Dirichlet problem (0.3) is essentially different in the two parameter ranges $\alpha \leq -1$ and $\alpha > -1$ as seen by considering the $\alpha$-harmonic function $u(z) = (\text{Im} z)^{\alpha+1}$ (a slightly more interesting example when $\alpha = -1$ is provided by $u(z) = \log \text{Im} z$); for $\alpha > -1$ this solution vanishes at the boundary, showing that solutions are not unique without additional constraints. Conversely, when $\alpha \leq -1$, $u$ is clearly singular on the boundary and thus not an eligible solution for any boundary data in $\mathcal{S}'$. This is symptomatic; when $\alpha \leq -1$ there are in fact no temperate solutions to the Dirichlet problem when the boundary data has compact support, see Theorem 1.6. We shall therefore restrict our attention mainly to the parameter range $\alpha > -1$. In this case, it turns out that the “homogeneous problem”, i.e. when $f \equiv 0$ in (0.3), has many solutions that are of polynomial growth. We call such solutions quasi-polynomial $\alpha$-harmonic functions. In Section 1 we give a precise characterization of this class and show that its elements $u$ are the only solutions to the homogeneous problem such that $u_y$ is of polynomial growth. Based on these results we establish uniqueness theorems for (0.3) at the end of Section 1.
In Section 2 we show that the Poisson kernel for the $\alpha$-harmonic Dirichlet problem is the function
$$\mathcal{P}_\alpha(z) = \frac{2^\alpha e^{i\alpha\pi/2}}{\pi} \cdot \frac{(\text{Im } z)^{\alpha+1}}{z^{\alpha+1}}, \quad z \in \mathbb{C}_+,$$
where $\alpha > -1$. By this we mean that $u(z) = \int \mathcal{P}_\alpha(z-t) f(t) dt$ is a solution of (0.3) if, say, $f \in L^\infty$ (so that the convolution is well defined for all $z \in \mathbb{C}_+$). This is established in Theorem 2.2. Clearly, the convolution is well defined for other classes than $L^\infty$, and the next task is to lift the above result to a more general setting. With $\mathcal{P}_{\alpha,y}$ defined in accordance with (0.4), it turns out that $\mathcal{F}(\mathcal{P}_{\alpha,y})$ is not smooth at the origin (Theorem 2.4), and hence the classical definition by Schwartz [24] of how to convolve with elements in $\mathcal{S}'$ does not apply. We elaborate more on this issue in Section 3, where we show how to overcome this obstruction by instead using the so-called $\mathcal{S}'$-convolution proposed by Hirata and Ogata [16]. We also determine the optimal class $w^{\alpha+2}D'_L$ (see Definition 3.1) of tempered distributions $f$ for which the $\mathcal{S}'$-convolution $\mathcal{P}_{\alpha,y} * f$ is well defined for all $y > 0$.

Finally, in Section 4 we show that the Poisson integral $\mathcal{P}_\alpha[f] : x + iy \mapsto \mathcal{P}_{\alpha,y} * f(x)$ indeed solves the Dirichlet problem (0.3) for $f \in w^{\alpha+2}D'_L$ (Theorem 4.1). We also show that under appropriate growth constraints, any solution to (0.3) has this form modulo an $\alpha$-harmonic quasi polynomial (Theorem 4.5). Finally, we give conditions under which the Poisson integral is the unique solution (Theorem 4.6).

We end this introduction with some observations about the real part of $\Delta_\alpha$, that is $\text{Re } (\Delta_\alpha) = \frac{1}{2}(\Delta_\alpha + \Delta_\alpha^*)$. When $\alpha$ is a negative integer, then the equation $\text{Re } (\Delta_\alpha) u = 0$ is satisfied by the family of axially symmetric harmonic functions in $(2-\alpha)$-dimensional space, considered in a meridian plane. As a consequence, solutions to $\text{Re } (\Delta_\alpha) u = 0$ have historically (for general $\alpha \in \mathbb{R}$ and in $n$-dimensional half space) gone by the name generalized axially symmetric potentials, see the exposition by Weinstein [27]. The corresponding Dirichlet problem with distributional boundary values was studied in Wittsten [28]. This type of extension problem is known to be connected to the fractional Laplacian [9], and has also shown strong connections to hyperbolic Brownian motion with drift, with applications in for example risk theory [5, 21]. Similarly, we wish to point out that (at least formally), the $\alpha$-harmonic equation (0.2) is the generator of a hyperbolic Brownian motion with vertical drift $\alpha/2$, and non-conventional imaginary “horizontal” drift $i\alpha/2$, compare with the mentioned paper by Baldi, Casadio Tarabusi, Figà-Talamanca and Yor [5].

1. $\alpha$-harmonic functions of polynomial growth

1.1. General properties

We begin by establishing general properties of solutions of the $\alpha$-harmonic equation $\Delta_\alpha u = 0$ in $\mathbb{C}_+$. We first remark that the operator $\Delta_\alpha$ is (non-strictly) elliptic in the upper half plane and therefore regularizing in the interior. In this context we think of
\( \Delta_\alpha \) as a second order differential operator in the variables \( x \) and \( y \),
\[
\Delta_\alpha = \frac{1}{4} \left( y^{-\alpha} (\partial_x^2 + \partial_y^2) - i\alpha y^{-\alpha-1} \partial_x - \alpha y^{-\alpha-1} \partial_y \right),
\]
and identify an \( \alpha \)-harmonic function \( u \) with a function of \( (x, y) \in \mathbb{R} \times (0, \infty) \), denoted \( u(x, y) \) for convenience. If \( X \) is a subset of \( \mathbb{R}^n \) we let \( \mathcal{E}_c^\infty(X) \) denote the set of elements in \( \mathcal{E}_c^\infty(\mathbb{R}^n) \) with support contained in \( X \). Similarly, we let \( \mathcal{E}^\prime(X) \) denote the set of distributions in \( \mathcal{D}'(\mathbb{R}^n) \) with support contained in \( X \). When \( n = 1 \) and there is no ambiguity we will usually omit \( \mathbb{R} \) from the notation. We let \( \mathbb{R}_+ = (0, \infty) \) and \( \mathbb{R}_- = (-\infty, 0) \) denote the positive and negative half axis, respectively, while \( \mathbb{R}_+^2 \) denotes the half space \( \{(x, y) \in \mathbb{R}^2 : y > 0 \} \).

**Proposition 1.1.** Let \( \alpha \in \mathbb{R} \). Let \( u \) be a solution to \( \Delta_\alpha u = 0 \) in \( \mathbb{C}_+ \equiv \mathbb{R}_+^2 \) in the sense of distribution theory. Then \( u \in \mathcal{E}_c^\infty(\mathbb{C}_+) \) and \( u \) solves \( \Delta_\alpha u = 0 \) in \( \mathbb{C}_+ \) also in the classical sense, so \( u \) is \( \alpha \)-harmonic.

**Proof.** Let \( Y \subset X \subset \mathbb{C}_+ \) be \( \mathcal{C}_1^\infty \) domains. (We write \( Y \subset X \) when \( Y \) is compact and contained in \( X \).) Choose a cutoff function \( \psi \in \mathcal{E}_c^\infty(X) \) such that \( \psi \equiv 1 \) in a neighborhood of \( Y \), and set \( v = \psi u \). Then \( v \in \mathcal{E}^\prime(X) \) and \( \Delta_\alpha v = 0 \) in \( Y \) since \( v = u \) in \( Y \). Moreover, \( \Delta_\alpha \) (restricted to \( X \)) is strictly elliptic in \( X \) and we can find a properly supported parametrix \( Q_\alpha \) of \( \Delta_\alpha \), that is, \( Q_\alpha \in \Psi^{-2}(X) \) is a properly supported pseudodifferential operator (of order \(-2\)) satisfying
\[
\Delta_\alpha \circ Q_\alpha \equiv Q_\alpha \circ \Delta_\alpha \equiv I \mod \Psi^{-\infty}(X),
\]
see Hörmander [18, Theorem 18.1.24]. Here \( I \) is the identity operator \( Iu = u \) for \( u \in \mathcal{D}'(X) \), and \( \Psi^{-\infty}(X) \) is the space of smoothing operators \( \mathcal{E}^\prime(X) \rightarrow \mathcal{E}_c^\infty(X) \). In particular, \( v \equiv Q_\alpha \circ \Delta_\alpha v \mod \mathcal{E}_c^\infty(X) \). Note also that \( Q_\alpha \) is pseudo-local, that is, \( \text{sing supp}(Q_\alpha \circ \Delta_\alpha v) \subset \text{sing supp}(\Delta_\alpha v) \), see for example Hörmander [18, Proposition 18.1.26] and Hörmander [17, Definition 8.1.2]. This gives
\[
Y \cap \text{sing supp } v = Y \cap \text{sing supp } (Q_\alpha \circ \Delta_\alpha v) \subset Y \cap \text{sing supp } (\Delta_\alpha v) = \emptyset.
\]
Now \( u = v \) in \( Y \) so \( u \in \mathcal{E}_c^\infty(Y) \). Since \( Y \) was arbitrary, we have \( u \in \mathcal{E}_c^\infty(\mathbb{C}_+) \) so \( u \) solves \( \Delta_\alpha u = 0 \) in the classical sense, and the proof is complete.

Next we remark that the class of \( \alpha \)-harmonic functions is invariant under real translations and dilations.

**Proposition 1.2.** Let \( \alpha \in \mathbb{R} \) and let \( u \) be \( \alpha \)-harmonic in \( \mathbb{C}_+ \). Let \( r > 0 \), \( t \in \mathbb{R} \) be given and set \( v(z) = u(rz + t) \). Then \( \Delta_\alpha v = 0 \) in \( \mathbb{C}_+ \).

**Proof.** If \( z \in \mathbb{C}_+ \) then \( rz + t \in \mathbb{C}_+ \) when \( r > 0 \) and \( t \in \mathbb{R} \). Hence, \( \Delta_\alpha u(rz + t) = 0 \). Noting that \( \Delta_\alpha = y^{-\alpha} \Delta - i\alpha 2^{-1} y^{-\alpha-1} \partial_t \), differentiation gives
\[
\Delta_\alpha v(z) = y^{-\alpha} r^2 \Delta u(rz + t) - \frac{i\alpha}{2} y^{-\alpha-1} r \partial_t u(rz + t)
= \frac{r^{\alpha+2} (ry)^{-\alpha} \Delta u(rz + t) - \frac{i\alpha}{2} (ry)^{-\alpha-1} \partial_t u(rz + t))}{r^{\alpha+2} \Delta_\alpha u(rz + t) = 0},
\]
which completes the proof.
If $y_0 > 0$ we say that $u$ is of polynomial growth in $iy_0 + C_+ = \{ z : \text{Im} z > y_0 \}$ if there are constants $C$ and $N$ (which may depend on $y_0$) such that

$$|u(\zeta)| \leq C(1 + |\zeta|)^N, \quad \text{Im} \zeta > y_0.$$ 

**Proposition 1.3.** Let $\alpha \in \mathbb{R}$ and let $Y \Subset X \Subset C_+$ be open, bounded sets with $Y$ having locally Lipschitz boundary. Then for any $m, n \in \mathbb{N}$ there exists a constant $C$ (depending also on $\alpha$, $X$ and $Y$) such that

$$\| \partial_x^m \partial_y^n u \|_{L^\infty(Y)} \leq C \| u \|_{L^\infty(\partial X)}$$

for all $\alpha$-harmonic functions $u$ in $C_+$. Moreover, suppose that $u$ has polynomial growth in $iy_0 + C_+$ for all $y_0 > 0$. Then the same assertion is true for $\partial_x^m \partial_y^n u$.

**Proof.** By the maximum principle (see for example Theorem 1 in Evans [12, Section 6.4]), it follows that $\|u\|_{L^2(Y)} \leq C_1 \|u\|_{L^\infty(\partial X)}$ for some $C_1 \equiv C_1(X)$. Next, given any $k \in \mathbb{N}$, Theorem 2 in Section 6.3 of the same book yields the existence of $C_2 \equiv C_2(\alpha, k, X, Y)$ such that

$$\| \partial_x^m \partial_y^n u \|_{H^{k+2}(Y)} \leq C_2 \| u \|_{L^2(X)},$$

where we use standard notation for the Sobolev spaces $H^k(X) = W^{k,2}(X)$. Finally, by the Sobolev embedding theorem we have $\|u\|_{C^k(\mathbb{R})} \leq C_3 \|u\|_{H^{k+2}(Y)}$ for some constant $C_3 \equiv C_3(k, Y)$, see Adams and Fournier [1, Theorem 4.12]. Combining these three inequalities yields (1.1).

Now let $y_0 > 0$ be arbitrary. Then by assumption there are constants $C_0$ and $N_0$ such that

$$|u(z)| \leq C_0(1 + |z|)^{N_0}, \quad \text{Im} z > y_0/2.$$  \hspace{1cm} (1.2)

Fix $\zeta = \xi + i\eta$ with $\eta > y_0$, and denote by $D_\zeta$ the disc with center $\zeta$ and radius $\eta/2$. Clearly $D_\zeta \subset \{ z : \text{Im} z > y_0/2 \}$, so by (1.2) we have

$$\sup_{D_\zeta} |u| \leq \sup_{z \in \partial D_\zeta} C_0(1 + |z|)^{N_0} \leq C_0(1 + |\zeta| + \eta/2)^{N_0} \leq \tilde{C}_0(1 + |\zeta|)^{N_0},$$

where $\tilde{C}_0 = C_0(3/2)^{N_0}$. Let $C$ be the bound for the map in (1.1) with $X = D_\zeta$ and $Y$ being any small disc containing $i$. Set $v(z) = u(\xi + \eta z)$. Then $\Delta_\zeta v = 0$ in $C_+$ by Proposition 1.2 and it is easily seen that the values of $u$ in $D_\zeta$ correspond to those of $v$ in $D_\zeta$. Moreover, $\partial_x^m \partial_y^n v(i) = \eta^{-m-n} \partial_x^m \partial_y^n u(\zeta)$, which gives

$$|\partial_x^m \partial_y^n u(\zeta)| = \eta^{-m-n} |\partial_x^m \partial_y^n v(i)| \leq y_0^{m-n} C \sup_{\partial D_\zeta} |v| = y_0^{m-n} C \sup_{\partial D_\zeta} |u|.$$

Hence, for all $\zeta \in C_+$ with $\text{Im} \zeta > y_0$ we have

$$|\partial_x^m \partial_y^n u(\zeta)| \leq y_0^{m-n} C \tilde{C}_0(1 + |\zeta|)^{N_0}$$

so $\partial_x^m \partial_y^n u$ has polynomial growth in $iy_0 + C_+$. Since $y_0 > 0$ was arbitrary, this completes the proof.
1.2. Fourier transforms of α-harmonic functions

We define the Fourier transform $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ by

$$\mathcal{F}\phi(\xi) = \hat{\phi}(\xi) = \int e^{-ix\xi}\phi(x)dx, \quad \phi \in \mathscr{S},$$

and extend the definition to $\mathscr{S}'$ by duality. The inverse will be denoted by $\mathcal{F}^{-1}$. Note that if $v \in \mathscr{C}^\infty$ has polynomial growth, then $v$ defines a tempered distribution (which we shall also denote by $v$) through $\phi \mapsto \int v(x)\phi(x)dx$. In particular, if $u \in \mathscr{C}^\infty(\mathbb{C}_+)$ has polynomial growth in $iy_0 + C_+$, then $|u(\zeta)| \leq C(1 + |\zeta|)^m$ when $\text{Im} \, \zeta > y_0$ for some constants $C$ and $m$. With $u_y$ defined in accordance with (0.4) it follows that $y \mapsto u_y$ defines a one-parameter family of tempered distributions for $y > y_0$ such that

$$|\langle u_y, \phi \rangle| \leq C(1 + y)^m \sup_x (1 + |x|)^{m+2} |\phi(x)|,$$  

(1.3)

where $C_1 = \int (1 + |x|)^{-2}dx$. Note that the supremum can be estimated by a semi-norm of $\phi$ in $\mathscr{S}$. Moreover, since $\langle \hat{u}_y, \phi \rangle = \langle u_y, \hat{\phi} \rangle$ and the Fourier transform is continuous in $\mathscr{S}$, it follows that $\hat{u}_y$ satisfies a similar estimate. In fact, we have that

$$(1 + |x|)^{m+2} |\hat{\phi}(x)| = \sum_{k=0}^{m+2} (m+2)_{m+k} |x^k \hat{\phi}(x)|,$$

where a straightforward estimation shows that $|x^k \hat{\phi}(x)| \leq C_1 \sup_\xi (1 + |\xi|)^2 |\phi^{(k)}(\xi)|$ with $C_1$ as above. Hence, an application of (1.3) gives

$$|\langle \hat{u}_y, \phi \rangle| \leq C(1 + y)^m \sum_{k=0}^{m+2} \sup_\xi (1 + |\xi|)^2 |\phi^{(k)}(\xi)|$$

(1.4)

for some new constant $C$.

We can now describe the Fourier transform of an $\alpha$-harmonic function of polynomial growth. To this end, we introduce the function

$$\gamma_\alpha(\xi, y) = \int_c^y t^\alpha e^{-2t\xi}dt$$

(1.5)

where we let $c = 1$ for $\alpha \leq -1$ and $c = 0$ else. (Of course, we use the standard convention $\int_c^y = -\int_y^c$ for $y < c$.)

**Lemma 1.4.** For $\alpha > -1$ we have $\gamma_\alpha(\cdot, y) \to 0$ in $\mathscr{C}^\infty$ as $y \to 0$. For $\alpha \leq -1$ we have $(\gamma_\alpha(\cdot, y))^{-1} \to 0$ in $\mathscr{C}^\infty$ as $y \to 0$. Moreover, for any $\alpha \in \mathbb{R}$ the limit $\gamma_\alpha(\xi, \infty)$ defines a $\mathscr{C}^\infty$ function of $\xi \in \mathbb{R}_+$.

**Proof.** If $\alpha > -1$ then the integral in (1.5) tends to 0 uniformly on compacts as $y \to 0$. Differentiating the integral with respect to $\xi$ increases the rate of convergence, so it is clear that $\gamma_\alpha(\cdot, y) \to 0$ in $\mathscr{C}^\infty$ as $y \to 0$. If $\xi > 0$ then $\int_c^\infty t^{\alpha+k} e^{-2t\xi}dt$ is absolutely convergent for any $k \in \mathbb{N}$, so the last part is also immediate.

Suppose therefore that $\alpha \leq -1$ and let $r > 0$ be given. For $|\xi| < r$ and $0 < y < 1$ we then have

$$|\gamma_\alpha(\xi, y)| \geq \int_y^1 t^\alpha e^{-2t\xi}dt \geq e^{-2r} \int_y^1 t^\alpha dt,$$

(1.6)
which shows that \((\gamma_\alpha(\cdot, y))^{-1} \to 0\) uniformly on compacts in \(\mathbb{R}\) as \(y \to 0\). We need to prove the corresponding statement for its derivatives. By induction, it is easy to see that 

\[
\partial^k_{\xi} (\gamma_\alpha(\xi, y))^{-1}
\]

can be written as a linear combination of terms of the form

\[
\frac{\partial^{\ell_1}_{\xi} \gamma_\alpha(\xi, y) \cdots \partial^{\ell_k}_{\xi} \gamma_\alpha(\xi, y)}{(\gamma_\alpha(\xi, y))^{k+1}}
\]

where \(\ell_1, \ldots, \ell_k\) are natural numbers that sum up to \(k\). Moreover, for each \(1 \leq j \leq k\) we have

\[
|\partial^j_{\xi} \gamma_\alpha(\xi, y)| = 2^{\ell_j} \int_y^1 t^{\alpha + \ell_j} e^{-2\xi t} dt \leq 2^{\ell_j} \int_y^1 t^\alpha e^{-2\xi t} dt = 2^{\ell_j} |\gamma_\alpha(\xi, y)|.
\]

Hence there is a constant \(C_k\) such that

\[
|\partial^k_{\xi} (\gamma_\alpha(\xi, y))^{-1}| \leq C_k |(\gamma_\alpha(\xi, y))^{-1}|,
\]

so the desired statement follows by (1.6).

**Proposition 1.5.** Let \(\alpha \in \mathbb{R}\). Let \(u\) be \(\alpha\)-harmonic in \(\mathbb{C}_+\) and assume that \(u\) is of polynomial growth in \(iy_0 + \mathbb{C}_+\) for all \(y_0 > 0\). Then there are distributions \(v\) and \(w\) in \(\mathcal{D}'\), where \(\text{supp } v \subset [0, \infty)\), such that

\[
\hat{u}_y = e^{\langle \cdot, y \rangle} \gamma_\alpha(\cdot, y)v + e^{\langle \cdot, y \rangle} w \tag{1.7}
\]

in \(\mathcal{D}'\). Moreover, on \(\mathbb{R}_+\) we have \(w = -\gamma_\alpha(\cdot, \infty)v\).

Before the proof, we point out that the restriction to functions with polynomial growth is not necessary. Working with the classical Paley-Wiener-Schwartz theorem, one can (by using a suitably generalized version on Proposition 1.3) deduce a similar statement for functions of exponential type, and by using the results in Carlsson and Wittsten [11] this can be further relaxed to functions of order less than 2. The latter reference also contains structure results for \(v\). However, since this will not be needed for the purposes of this paper, it has been omitted for simplicity.

**Proof.** By assumption, \(\partial y^{-\alpha} \partial u(z) = 0\) when \(z = x + iy \in \mathbb{C}_+\). Hence, \(x + iy \mapsto y^{-\alpha} \partial u(x + iy)\) is holomorphic in the upper half plane, and by Proposition 1.3 it is of polynomial growth in \(iy_0 + \mathbb{C}_+\) for all \(y_0 > 0\). Thus, by virtue of the Paley-Wiener-Schwartz theorem there is a distribution \(\tilde{v} \in \mathcal{D}'\) with \(\text{supp } \tilde{v} \subset (-\infty, 0)\) such that \(e^{\langle \cdot, y \rangle} \tilde{v} \in \mathcal{D}'\) and

\[
y^{-\alpha} \partial u(x + iy) = \mathcal{F}(e^{\langle \cdot, y \rangle} \tilde{v})(x) \tag{1.8}
\]

for all \(y > 0\), see Hörmander [17, Theorem 7.4.3].

Next, let \(\phi \in \mathcal{C}_c^\infty\) and consider the map

\[
y \mapsto \langle u(\cdot + iy), \mathcal{F}(e^{-\langle \cdot, y \rangle} \phi)\rangle, \quad y > 0,
\]

where we for the moment write \(u(\cdot + iy)\) instead of \(u_y\) for clarity. We claim that this map is \(\mathcal{C}_c^\infty\) and

\[
\partial_y \langle u(\cdot + iy), \mathcal{F}(e^{-\langle \cdot, y \rangle} \phi)\rangle = 2i \langle \partial u(\cdot + iy), \mathcal{F}(e^{-\langle \cdot, y \rangle} \phi)\rangle. \tag{1.9}
\]
Indeed, \( \mathcal{F}(e^{-iy}\phi) \in \mathcal{F} \) for all \( y \), and it is easy to see that \( \partial_y \mathcal{F}(e^{-iy}\phi)(\xi) = \mathcal{F}(\partial_y(e^{-iy}\phi))(\xi) \). Using Taylor’s formula and Proposition 1.3 to estimate the remainder, one can also check that \( \partial_y^\alpha(u(\cdot + iy), \psi) = \langle \partial_y^\alpha u(\cdot, iy), \psi \rangle \) for \( \psi \in \mathcal{F} \) and \( y > 0 \) (compare with the proof of Hörmander [17, Theorem 2.1.3]). By the chain rule we thus have

\[
\partial_y(u(\cdot + iy), \mathcal{F}(e^{-iy}\phi)) = (\partial_y u(\cdot, iy), \mathcal{F}(e^{-iy}\phi)) + (u(\cdot + iy), \mathcal{F}(\partial_y(e^{-iy}\phi))).
\]

Consider the second term and note that

\[
\mathcal{F}(\partial_y(e^{-iy}\phi))(x) = -i\partial_x \mathcal{F}(e^{-iy}\phi)(x).
\]

Integrating by parts and using \( 2i\partial_x = i\partial_x u + \partial_y u \) proves the claim.

For \( \phi \in \mathcal{C}_\infty \), a combination of (1.8) and (1.9) gives

\[
\partial_y(e^{-iy}\hat{u}_y, \phi) = 2i(y^\alpha \mathcal{F}(e^{iy}\tilde{v}), \mathcal{F}(e^{-iy}\phi)).
\]

In view of the Fourier inversion formula, this implies that

\[
\partial_y(e^{-iy}\hat{u}_y, \phi) = \langle v, y^\alpha e^{-2iy}\phi \rangle,
\]

where \( v = 4\pi iy \hat{\phi} \) satisfies \( \sup v \subset [0, \infty) \). For \( y > 0 \), the expression on the right is the derivative of \( \langle v, \gamma_{\alpha}(\cdot, y) \rangle \) (see Hörmander [17, Theorem 2.1.3]). Hence,

\[
(e^{-iy}\hat{u}_y, \phi) = \langle v, \gamma_{\alpha}(\cdot, y) \rangle + w_\phi
\]

(1.10)

for some constant \( w_\phi \) (that is, independent of \( y \)). It is clear that the map \( \phi \mapsto w_\phi \) is a continuous linear form on \( \mathcal{C}_\infty \). In fact, if \( \phi_j \to 0 \) in \( \mathcal{C}_\infty \) as \( j \to \infty \), then \( e^{-iy}\phi_j \) and \( \phi_j\gamma_{\alpha}(\cdot, y) \) tend to zero in \( \mathcal{C}_\infty \) as \( j \to \infty \). Since \( \hat{u}_y \) and \( v \) are continuous forms on \( \mathcal{C}_\infty \), the claim follows. Since multiplication by \( e^{iy}\phi \) is continuous and invertible on \( \mathcal{C}_\infty \), (1.7) is a direct consequence of (1.10).

For the second part, let \( \phi \in \mathcal{C}_\infty \) have support in \( \mathbb{R}_+ \). Since \( u \) has polynomial growth in \( iy_0 + \mathbb{C}_+ \) we can by (1.4) find \( C \) and \( m \) such that

\[
|\langle e^{-iy}\hat{u}_y, \phi \rangle| \leq C(1 + y)^m \sum_{k=0}^{m+2} \sup_{\xi} (1 + |\xi|)^2 |\partial^k_\xi(e^{-y\xi}\phi(\xi))|, \quad y > y_0.
\]

It follows that the left-hand side of (1.10) tends to 0 as \( y \to \infty \). On the other hand, standard analysis shows that \( \gamma_{\alpha}(\cdot, y)\phi \) converges to \( \gamma_{\alpha}(\cdot, \infty)\phi \) as \( y \to \infty \), by which it follows that \( 0 = \langle v, \phi \gamma_{\alpha}(\cdot, \infty) \rangle + \langle w, \phi \rangle \), as desired.

The next result shows that the restriction \( \alpha > -1 \) in the parameter range is essential for a satisfactory existence theory for the \( \alpha \)-harmonic Dirichlet problem (0.3).

**Theorem 1.6.** Let \( \alpha \leq -1 \). Let \( u \) be \( \alpha \)-harmonic in \( \mathbb{C}_+ \) and assume that \( u \) is of polynomial growth in \( iy_0 + \mathbb{C}_+ \) for all \( y_0 > 0 \). If \( u_y \to f \) in \( \mathcal{F} \) as \( y \to 0 \), then \( \supp f \subset (-\infty, 0] \). In particular, \( f \) cannot have compact support unless \( f \equiv 0 \).
Proof. If \( f \in \mathcal{S}' \) has compact support then \( \hat{f} \) is real analytic, see Hörmander [17, Theorem 7.1.14]. Hence the second part is an immediate consequence of the first part and the fact that real analytic functions have isolated zeros.

To prove the first part, it suffices to show that \( v \) in Proposition 1.5 cannot have support in \( \mathbb{R}_+ \). Indeed, if \( \text{supp} \ v \cap \mathbb{R}_+ \neq \emptyset \) then the final identity of that proposition shows that \( \text{supp} \ w \cap \mathbb{R}_+ \neq \emptyset \), and then (1.7) implies the same for \( \hat{u}_y \). Since our assumptions imply that \( \hat{u}_y \to \hat{f} \) in \( \mathcal{S}' \) as \( y \to 0 \) we obtain \( \text{supp} \ \hat{f} \cap \mathbb{R}_+ = \emptyset \).

Now, let \( \phi \in \mathcal{E}^\infty_c \) have support in \( \mathbb{R}_+ \) and note that (1.7) implies that
\[
\langle e^{iy} v, \phi \rangle = \langle \hat{u}_y, \phi(\gamma(\cdot, y))^{-1} \rangle - \langle e^{iy} w, \phi(\gamma(\cdot, y))^{-1} \rangle.
\]
By Lemma 1.4 we have that \( \phi(\gamma(\cdot, y))^{-1} \to 0 \) in \( \mathcal{E}^\infty_c \) as \( y \to 0 \). According to Hörmander [17, Theorem 2.1.8], the limit of the right-hand side in (1.11) equals 0, and hence \( \langle v, \phi \rangle = 0 \), as desired.

1.3. Fourier transforms for the case \( \alpha > -1 \)

We shall henceforth only treat the case \( \alpha > -1 \). The representation formula for an \( \alpha \)-harmonic function \( u \) provided by Proposition 1.5 can then be improved. To simplify the statement we introduce the auxiliary function
\[
H_\alpha(\xi, y) = \begin{cases} (\Gamma(\alpha + 1))^{-1} \int_{2y}^\infty t^\alpha e^{-t} dt, & \xi > 0, \\ 1, & \xi \leq 0, \end{cases}
\]
defined for \( y > 0 \), where \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \) is the Gamma function. In particular, \( H_\alpha(\cdot, y) \) is continuous since \( \int_{2y}^\infty t^\alpha e^{-t} dt \to \Gamma(\alpha + 1) \) as \( \xi \to 0 \). It will later be shown that the product \( e^{iy} H_\alpha(\cdot, y) \) coincides with the Fourier transform of the kernel function \( \mathcal{P}_\alpha,q \) mentioned in the introduction, see Theorem 2.4 below.

**Theorem 1.7.** Let \( \alpha > -1 \) and assume that \( u \) is of polynomial growth in \( iy_0 + \mathcal{C}_+ \) for all \( y_0 > 0 \). Let \( f \in \mathcal{S}' \) be such that \( f \in L^1_{\text{loc}} \) and suppose that \( u \to f \) in \( \mathcal{S}' \) as \( y \to 0 \). Then \( H_\alpha(\cdot, y) \hat{f} \in L^1_{\text{loc}} \) and there is a \( q \in \mathcal{D}' \) with \( \text{supp} \ q \subset \{0\} \) such that
\[
\hat{u}_y = e^{iy} H_\alpha(\cdot, y) \hat{f} + e^{iy} \gamma(\cdot, y) q
\]
in \( \mathcal{D}' \).

**Proof.** First note that since \( H_\alpha(\cdot, y) \) is uniformly bounded on compacts, it follows that we have \( H_\alpha(\cdot, y) \hat{f} \in L^1_{\text{loc}} \) whenever \( \hat{f} \in L^1_{\text{loc}} \).

Next, let \( \phi \in \mathcal{E}^\infty_c \) be arbitrary and recall the representation (1.7) from Proposition 1.5. Since \( \phi \gamma(\cdot, y) \to 0 \) in \( \mathcal{E}^\infty_c \) as \( y \to 0 \), we immediately get that
\[
\langle \hat{f}, \phi \rangle = \lim_{y \to 0} \langle \hat{u}_y, \phi \rangle = \lim_{y \to 0} (\langle v, e^{iy} \gamma(\cdot, y) \phi \rangle + \langle w, e^{iy} \phi \rangle) = \langle w, \phi \rangle
\]
so \( w = \hat{f} \). Proposition 1.5 also gives that
\[
v = -w(\gamma(\cdot, \infty))^{-1} = -\hat{f}(\gamma(\cdot, \infty))^{-1}
\]
on $\mathbb{R}_+$. Let $1_{\Omega}$ denote the characteristic function of a measurable set $\Omega$, and set $\hat{v} = -1_{\mathbb{R}_+} (\gamma_\alpha(\cdot, \infty))^{-1} \hat{f}$. Since $\hat{f} \in L^1_{\text{loc}}$ and $\gamma_\alpha(\xi, \infty) \to \infty$ as $\xi \to 0^+$ it follows that $\hat{v}$ exists as a function in $L^1_{\text{loc}}$. Moreover, $\text{supp} \, v \subset [0, \infty)$ so if we set $q = v - \hat{v}$ then $q \in D'$ and $\text{supp} \, q \subset \{0\}$.

Now, let $h_\alpha(\cdot, y)$ be a function that equals 1 on $\mathbb{R}_-$ and $1 - \gamma_\alpha(\cdot, y)(\gamma_\alpha(\cdot, \infty))^{-1}$ on $\mathbb{R}_+$. By (1.7) and the above we have that

$$
\hat{u}_y = e^{(\cdot, y)} \gamma_\alpha(\cdot, y) q + e^{(\cdot, y)} h_\alpha(\cdot, y) \hat{f}.
$$

The result therefore follows if we show that $h_\alpha(\cdot, y)$ can be extended to a continuous function coinciding with $H_\alpha(\cdot, y)$. For $\xi < 0$ we have $h_\alpha(\xi, y) = H_\alpha(\xi, y)$ by definition. For $\xi > 0$ we have

$$
h_\alpha(\xi, y) = 1 - \int_0^\infty \frac{t^\alpha e^{-2t\xi}}{t^\alpha e^{-2t\xi}} dt = 1 - \int_0^\infty \frac{t^\alpha e^{-t\xi}}{t^\alpha e^{-t\xi}} dt = \int_0^\infty \frac{t^\alpha e^{-t\xi}}{t^\alpha e^{-t\xi}} dt
$$

so $h_\alpha(\xi, y) = H_\alpha(\xi, y)$ also for $\xi > 0$ since $\int_0^\infty t^\alpha e^{-t\xi} dt = \Gamma(\alpha+1)$. The last identity also shows that $h_\alpha(\xi, y) \to 1$ as $\xi \to 0$, which completes the proof.

We remark that the condition $\hat{f} \in L^1_{\text{loc}}$ certainly can be relaxed, since the only issue in the above proof was caused by lack of smoothness of $h_\alpha$ at 0. We will not pursue this direction further here.

### 1.4. $\alpha$-harmonic quasi-polynomials

We will now clarify the role of the distribution $q$ appearing in Theorem 1.7, which can be thought of as incorporating all solutions to the homogeneous Dirichlet problem, that is, the $\alpha$-harmonic functions vanishing on the boundary.

**Definition 1.8.** An $\alpha$-harmonic function $u$ consisting of a finite linear combination of terms of the form $y^{\alpha+1+j} z^k$ (where $j, k \in \mathbb{N}$ and $z = x + iy$) is said to be an $\alpha$-harmonic quasi-polynomial.

It is easy to construct $\alpha$-harmonic quasi-polynomials. Indeed, $u$ is by definition $\alpha$-harmonic in $\mathbb{C}_+$ precisely when $x + iy \mapsto y^{-\alpha} \partial u(x + iy)$ is holomorphic in $\mathbb{C}_+$. If $p(z)$ is an analytic polynomial, then the solution to the equation

$$
\partial u(z) = y^\alpha p(z), \quad z = x + iy \in \mathbb{C}_+,
$$

is therefore $\alpha$-harmonic. It is easily seen that one solution to (1.13) is given by

$$
u(z) = \frac{2 i y^{\alpha+1} p(z)}{\alpha + 1} - \frac{(2i)^2 y^{\alpha+2} \partial p(z)}{(\alpha + 1)(\alpha + 2)} + \cdots
$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2i)^m y^{\alpha+1+m} \partial^m p(z)}{(\alpha + 1) \cdots (\alpha + 1 + m)}, \tag{1.14}
$$

since the sum is finite as $\partial^m p \equiv 0$ for large enough $m$. By construction, $u$ is clearly quasi-polynomial. Note also that if the solution to (1.13) in addition is required to be continuous on $\mathbb{C}_+$ and zero on $\mathbb{R}$, then $u$ given by (1.14) is in fact the only solution.
Indeed, if $u_0$ is another such solution, then $u - u_0$ is conjugate analytic in $\mathbb{C}_+$ and vanishes at the boundary, so the claim follows by analytic continuation and the fact that conjugate analytic functions have isolated zeros, see e.g. Lang [20, Chapter XI, §1].

Remark. Note that (1.14) is $\alpha$-harmonic for all $\alpha \in \mathbb{R}$. However, when $\alpha \leq -1$ this function no longer has well defined boundary values in $\mathcal{S}'$ as $y \to 0$. The reader may compare with the discussion in the introduction concerning the function $x + iy \mapsto y^{\alpha+1}$, which is obtained as a special case of $\alpha$-harmonic quasi-polynomials by choosing $p$ in (1.14) to be an appropriate constant.

Proposition 1.9. Let $\alpha > -1$. Then the following are equivalent:

(i) $u$ satisfies the hypotheses of Theorem 1.7 with $f \equiv 0$.

(ii) There exists a distribution $q$ with $\text{supp} \ q = \{0\}$ such that

\[
  u(x + iy) = \mathcal{F}^{-1} \left( e^{i\langle \cdot, y \rangle} \gamma_\alpha(\cdot, y)q(\cdot) \right)(x)
\]  \hspace{1cm} (1.15)

in $\mathcal{D}'$.

(iii) $u$ is of the form (1.14).

Proof. The implication (iii) $\Rightarrow$ (i) is obvious, and Theorem 1.7 shows that (i) $\Rightarrow$ (ii).

It remains to establish that (ii) $\Rightarrow$ (iii). Note that by (1.15) we may write $u(z) = \frac{1}{2\pi} \mathcal{F}^{-1} \left( \mathcal{F} \gamma_\alpha(\cdot, y)q(\cdot) \right)(x)$ and $\partial = \frac{1}{2} (\partial_x - i\partial_y)$, so

\[
  \partial u(x + iy) = (2i)^{-1} \mathcal{F}^{-1} \left[ \left( \partial_y - \xi \right) (e^{y\xi} \gamma_\alpha(\xi, y)q(\xi)) \right](x)
\]  \hspace{1cm} (1.16)

in the sense of distribution theory (where we permit us to write $q(\xi)$ also for $q \in \mathcal{D}'$).

By the definition of $\gamma_\alpha$ (see (1.5)) we get

\[
  (\partial_y - \xi) (e^{y\xi} \gamma_\alpha(\xi, y)q(\xi)) = y^\alpha e^{-y\xi} q(\xi).
\]

Inserting this into (1.16) gives

\[
  \partial u(x + iy) = y^\alpha (2i)^{-1} \mathcal{F}^{-1} \left( e^{-\langle \cdot, y \rangle} q \right) = y^\alpha (4\pi i)^{-1} \langle q, e^{\langle \cdot, x + iy \rangle} \rangle,
\]  \hspace{1cm} (1.17)

where the expression in the last bracket is the Fourier-Laplace transform of $q$. Now, $q$ is a finite linear combination of derivatives of the Dirac measure at 0 since $\text{supp} \ q = \{0\}$, see Hörmander [17, Theorem 2.3.4]. In view of (1.17) we thus conclude that $\partial u(z) = (\text{Im} \ z)^n p(z)$ for some analytic polynomial $p$ of degree equal to the order of $q$. In particular, $u$ is the unique solution of (1.13) and thus given by (1.14), as explained there. This completes the proof.

In particular, we have the following uniqueness result:

Corollary 1.10. Let $\alpha > -1$. Let $u$ be $\alpha$-harmonic in $\mathbb{C}_+$, and assume that

(i) $z \mapsto u(iy_0 + z)$ has polynomial growth for all $y_0 > 0$,
(ii) $u(x + iy)$ converges to 0 in $\mathcal{S}'$ as $y \to 0$, and

(iii) there is a $\delta > 0$ such that $\lim_{y\to\infty} y^{-\alpha-1}|u(x + iy)| = 0$ for all $|x| < \delta$.

Then $u \equiv 0$ in $\mathbb{C}_+$.  

**Proof.** The first two conditions combined with Proposition 1.9 implies that $u$ is of the form (1.14) for some analytic polynomial $p$ of degree $m$. By the binomial theorem we have

$$u(x + iy) = p(x) + y r_1(x) + \ldots + y^m r_m(x)$$

where the $r_j$'s are polynomials of degree $\leq m - j$. Inspecting (1.14) we conclude by similar reasoning that

$$u(x + iy) = 2^{i\alpha + 1} e^{i\alpha \pi/2} \frac{\Im z}{\pi} \cdot |z|^{\alpha+1}, \quad z \in \mathbb{C}_+.$$

for some new polynomials $R_j$. Since condition (iii) implies that

$$\lim_{y\to\infty} y^{-\alpha-1-k}|u(x + iy)| = 0, \quad |x| < \delta,$$

for all $k \in \mathbb{N}$, we may successively conclude that all the $R_j$'s must vanish identically (starting with $R_m$), which finally implies that $p$ must be identically zero. But then so is $u$ and the proof is complete. 

**Remark.** As evidenced by the harmonic function $x + iy \mapsto xy$ it is in general not possible to replace condition (iii) in Corollary 1.10 with the weaker assumption

(iii') $\lim_{y\to\infty} y^{-\alpha-1}|u(iy)| = 0$.

However, we have not found similar counterexamples when $\alpha \neq 0$, but also no proof of their nonexistence, so it is possible that the harmonic case $\alpha = 0$ is exceptional in this regard.

2. The $\alpha$-harmonic Poisson kernel

In this section we introduce the $\alpha$-harmonic Poisson kernel and analyze some of its properties.

**Definition 2.1.** Let $\alpha > -1$. Define the $\alpha$-harmonic Poisson kernel by

$$\mathcal{P}_\alpha(z) = 2^\alpha e^{i\alpha \pi / 2} \frac{\Im z^{\alpha+1}}{\pi z^{\alpha+1}}, \quad z \in \mathbb{C}_+.$$

Note that the classical Poisson kernel for the upper half plane is obtained for the parameter value $\alpha = 0$. For general $\alpha > -1$, the definition has merit in view of the following result.

**Theorem 2.2.** Let $\alpha > -1$. Then the $\alpha$-harmonic Poisson kernel $\mathcal{P}_\alpha$ is $\alpha$-harmonic in $\mathbb{C}_+$ and has the boundary limit $\lim_{y\to 0} \mathcal{P}_{\alpha,y} = \delta_0$ in $\mathcal{S}'$, where $\mathcal{P}_{\alpha,y}$ is defined in accordance with (0.4).
Proof. Let \( C_\alpha \) denote the normalizing constant in Definition 2.1, i.e. \( C_\alpha = 2^\alpha e^{i\alpha \pi/2}/\pi \).

A straightforward differentiation gives that
\[
\partial \mathcal{P}_\alpha(z) = C_\alpha \frac{\alpha + 1}{2i} \cdot \frac{(\text{Im } z)^\alpha}{z^{\alpha + 2}}, \quad z \in \mathbb{C}_+.
\]

This formula makes evident that the function \( z \mapsto (\text{Im } z)^{-\alpha}\partial \mathcal{P}_\alpha(z) \) is analytic, showing that \( \mathcal{P}_\alpha \) is \( \alpha \)-harmonic in \( \mathbb{C}_+ \).

We proceed to analyze the boundary limit of \( \mathcal{P}_\alpha \). Note that \( \mathcal{P}_{\alpha,y}(x) = \mathcal{P}_\alpha(x + iy) = y^{-1}\mathcal{P}_{\alpha,1}(x/y), \quad x \in \mathbb{R}, \) (2.1)

for \( y > 0 \). Since the function \( \mathcal{P}_{\alpha,1} \) clearly has bounded \( L^1(\mathbb{R}) \) norm when \( \alpha > -1 \), a standard construction of approximate identities ensures that \( \lim_{y \to 0} \mathcal{P}_{\alpha,y} = \delta_0 \) in \( \mathcal{S}' \), provided that
\[
\int_{-\infty}^{\infty} \mathcal{P}_{\alpha,1}(x)dx = 1, \tag{2.2}
\]

see e.g. Katznelson [19, Section VI.1.13], Hörmander [17, Theorem 1.3.2] or Carlsson [10] for a more thorough analysis. To see that (2.2) holds, let
\[
I_\alpha = \int_{-\infty}^{\infty} \frac{dx}{(x - i)(x + i)^{\alpha + 1}}
\]

for \( \alpha > -1 \). We need to show that \( I_\alpha = 1/C_\alpha \). Set \( f(z) = (z - i)^{-1}(z + i)^{-\alpha - 1} \). By using Cauchy’s residue theorem and a contour integral along the semi-circle \( |z| = R \) in the upper half plane, we obtain
\[
I_\alpha = 2\pi i \text{ Res } (f, i) = \frac{2\pi i}{(2i)^{\alpha + 1}} = C_\alpha^{-1}
\]

by letting \( R \to \infty \). This completes the proof. \( \square \)

We next show that \( \mathcal{P}_{\alpha,y} \in L^1 \).

Proposition 2.3. Let \( \alpha > -1 \). Then
\[
\| \mathcal{P}_{\alpha,y} \|_{L^1} = \frac{2^\alpha}{\sqrt{\pi}} \frac{\Gamma((\alpha + 1)/2)}{\Gamma(\alpha/2 + 1)}, \quad y > 0.
\]

Proof. By (2.1) we have that
\[
\| \mathcal{P}_{\alpha,y} \|_{L^1} = \| \mathcal{P}_{\alpha,1} \|_{L^1} = \frac{2^\alpha}{\pi} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^{\alpha/2 + 1}}
\]

for \( \alpha > -1 \). Moreover
\[
\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^{\alpha/2 + 1}} = \int_{0}^{\infty} \frac{y^{-1/2}dy}{(y + 1)^{\alpha/2 + 1}} = \frac{\Gamma((\alpha + 1)/2)\Gamma(\frac{1}{2})}{\Gamma(\alpha/2 + 1)}
\]

where the last formula is a special case of (1.1.20) in Andrews, Askey, and Roy [4]. Since \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \), the result follows. \( \square \)
For $f \in L^\infty$ we can thus define

$$P_\alpha[f] : z \mapsto P_\alpha,y \ast f(x) = \int P_\alpha(z-t)f(t)dt, \quad z = x + iy \in \mathbb{C}_+.$$ 

It is now a simple corollary of Theorem 2.2 and Proposition 2.3 to deduce that $P_\alpha[f]$ is a solution to the Dirichlet problem (0.3). However, we will return to this in greater generality in Corollary 4.2. We shall next calculate the Fourier transform $F(P_\alpha,y)$.

**Theorem 2.4.** Let $\alpha > -1$. Then the Fourier transform of the function $P_\alpha,y$ is given by

$$F(P_\alpha,y)(\xi) = e^{\gamma \xi} \text{ for } \xi \leq 0$$

and

$$F(P_\alpha,y)(\xi) = e^{\gamma \xi} \frac{\Gamma(\alpha + 1)}{\xi^{\alpha+1}} \int_0^\infty t^\alpha e^{-xt}dt$$

for $\xi > 0$.

Note that for $\alpha = 0$ we recover the Fourier transform of the classical Poisson kernel $P_{0,y}$, that is, $F(P_{0,y})(\xi) = e^{-\gamma |\xi|}$. A direct proof of the above formula can be based on the identity

$$\int_0^\infty t^\alpha e^{-zt}dt = \frac{\Gamma(\alpha + 1)}{z^{\alpha+1}}.$$ 

However, the indirect proof given below is shorter.

**Proof.** With the notation of Section 1.3, we must show that $F(P_\alpha,y)(\xi) = e^{\gamma \xi} H_\alpha(\xi, y)$ for $y > 0$ fixed but arbitrary. By Theorem 2.2, Theorem 1.7 applies to $u = P_\alpha$, and moreover, we have $F(P_\alpha,y) \to 1$ in $\mathscr{S}'$ as $y \to 0$. Thus

$$F(P_\alpha,y) = e^{\gamma \xi} H_\alpha(\xi, y) + e^{\gamma \xi} \gamma_\alpha(\xi, y)q$$

in $\mathscr{D}'$, where $\text{supp } q \subset \{0\}$. However, by Proposition 2.3 and the definition of $H_\alpha$, both $F(P_\alpha,y)$ and $e^{\gamma \xi} H_\alpha(\xi, y)$ are continuous functions, and hence $q \equiv 0$, as desired.

Note that Theorem 2.4 shows that the function $e^{\gamma \xi} H_\alpha(\xi, y)$ appearing in (1.12) is equal to $F(P_\alpha,y)$.

3. Weighted distribution spaces and the $\mathscr{S}'$-convolution

Our principal aim is to convolve $P_\alpha$ with a class of distributions $f$ as large as possible. Unfortunately, the convolution $f \ast v$ for $f \in \mathscr{S}'$, as originally set up by Schwartz [24], is only defined when $v \in \mathcal{O}'_C$, where $\mathcal{O}'_C$ is the space of rapidly decaying distributions. Since the Fourier transform is an isomorphism between $\mathcal{O}'_C$ and the space $\mathcal{O}'_M$ of slowly growing smooth functions (see Schwartz [24, Chapitre VII]) this is not possible in our case because the Fourier transform of $P_\alpha,y$ is not smooth at the origin, as we saw in Theorem 2.4. To overcome this obstruction we instead use the so-called $\mathscr{S}'$-convolution proposed by Hirata and Ogata [16] and later given an equivalent form by Shiraishi [25].

In subsection 3.1 we recall properties of certain weighted spaces of distributions. These spaces appear naturally in the context of Newtonian potentials of distributions, and we refer to Schwartz [24, Chapitre VI, §8] for details. They have subsequently
been studied by many authors, e.g. by Alvarez, Guzmán–Partida and Skórnik [3] to characterize the tempered distributions that are $\mathcal{F}'$-convolvable with the classical Poisson kernel for the half space, and by Alvarez, Guzmán–Partida and Pérez–Esteva [2] to study harmonic extensions of distributions. In subsection 3.2 we recall the definition of the $\mathcal{F}'$-convolution, as well as prove some auxiliary results of general nature, whereas subsection 3.3 is devoted to results for the particular kernel $\mathcal{P}_{a,y}$.

3.1. Weighted spaces of distributions

Given a differentiable function $\varphi$ on $\mathbb{R}$, we will let $d^m \varphi$ (and sometimes $\varphi^{(m)}$) denote its derivative of order $m$, since the usual symbol $\partial$ is reserved for complex differentiation. However, we still use $\partial_x$ and $\partial_y$ to denote partial derivatives of functions of several variables. Let $\mathcal{B}$ denote the vector space of smooth functions $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ such that all derivatives $d^m \varphi$ belong to $L^\infty$. We endow $\mathcal{B}$ with the topology induced by the seminorms $\varphi \mapsto \|d^m \varphi\|_{L^\infty}$. We will let $\tilde{\mathcal{B}}$ denote the closed subspace consisting of those elements $\varphi \in \mathcal{B}$ such that $d^m \varphi(x) \to 0$ as $|x| \to \infty$ for all $m \in \mathbb{N}$. The space $\mathcal{C}_c^\infty$ of compactly supported smooth functions is dense in $\tilde{\mathcal{B}}$, but $\mathcal{C}_c^\infty$ is not dense in $\mathcal{B}$.

We let $\mathcal{D}'_{L^1}$ denote the dual of $\mathcal{B}$. Since $\mathcal{C}_c^\infty$ is continuously embedded and dense in $\tilde{\mathcal{B}}$, it follows that $\mathcal{D}'_{L^1}$ is a space of distributions, in the sense that the restriction of an element $f \in \mathcal{D}'_{L^1}$ to $\mathcal{C}_c^\infty$ is in $\mathcal{D}'$, and conversely, an element of $\mathcal{D}'$ which is continuous with respect to the topology of $\mathcal{B}$ defines a unique extension to $\tilde{\mathcal{B}}$ which is in $\mathcal{D}'_{L^1}$. We will use the same notation for the distribution and its extension to $\mathcal{B}$. The dual of $\mathcal{B}$ is, in this sense, not a space of distributions. By Schwartz [24, Chapitre VI, Théorème XXV], a distribution $f$ belongs to $\mathcal{D}'_{L^1}$ if and only if $f$ can be represented as a finite sum

$$f = \sum_m d^m f_m, \quad f_m \in L^1,$$

where the derivatives are interpreted in the distributional sense. The distributions in $\mathcal{D}'_{L^1}$ are sometimes referred to as integrable distributions.

**Definition 3.1.** Set $w(x) = (1 + x^2)^{\frac{1}{2}}$ and let $\mu \in \mathbb{R}$. We define $w^\mu \mathcal{B}$ as

$$w^\mu \mathcal{B} = \{ \varphi \in \mathcal{C}_c^\infty : w^{-\mu} \varphi \in \mathcal{B} \}$$

with the topology induced by the map from $w^\mu \mathcal{B}$ to $\mathcal{B}$ given by $\phi \mapsto w^{-\mu} \phi$. The analogous definition applies also to $\mathcal{B}$. Moreover, we define the weighted space of distributions $w^\mu \mathcal{D}'_{L^1}$ as

$$w^\mu \mathcal{D}'_{L^1} = \{ f \in \mathcal{D}' : w^{-\mu} f \in \mathcal{D}'_{L^1} \}$$

with the topology induced by the map from $w^\mu \mathcal{D}'_{L^1}$ to $\mathcal{D}'_{L^1}$, given by $f \mapsto w^{-\mu} f$.

Clearly, $w^\mu \mathcal{D}'_{L^1}$ can (and will) be identified with the dual of $w^{-\mu} \mathcal{B}$ via the pairing

$$(f, \varphi)_{w^\mu \mathcal{D}'_{L^1}, w^{-\mu} \mathcal{B}} = (w^{-\mu} f, w^\mu \varphi)_{\mathcal{D}'_{L^1}, \mathcal{B}}. \quad (3.2)$$

It is well known that the derivatives of $w$ satisfies

$$|d^m w^\mu| \leq C_m w^{\mu-m}, \quad \mu \in \mathbb{R}, \quad (3.3)$$
where \( m \in \mathbb{N} \) and \( C_m > 0 \) is a constant. It follows that \( w^\mu \) is a so-called order function, that is, \( d^m w^\mu = O(w^\mu) \) for any \( m \in \mathbb{N} \). This fact combined with (3.1)–(3.2) can easily be used to show that \( w^\mu \mathcal{D}'_L \) admits the following useful representation:

\[
\mathcal{D}'_L \mathcal{D} \mathcal{F} = \left\{ f \in \mathcal{F} : f = \sum_m d^m f_m, \quad f_m \in L^1(w^{-\mu}) \right\}, \tag{3.4}
\]

where the summation is over a finite set (see also Alvarez et al. [2, Lemma 3.3] and [2, Remark 3.4]). In other words, for \( \varphi \in w^{-\mu} \mathcal{B} \), we have

\[
\langle f, \varphi \rangle_{w^{-\mu} \mathcal{D}'_L, w^{-\mu} \mathcal{B}} = \sum_m (-1)^m \int f_m(t) \varphi^{(m)}(t) dt. \tag{3.5}
\]

Here, the right-hand side makes sense also for \( \varphi \in w^{-\mu} \mathcal{B} \) and thus defines a continuous linear functional on \( w^{-\mu} \mathcal{B} \). For this reason, we will write \( \langle f, \varphi \rangle_{w^{-\mu} \mathcal{D}'_L, w^{-\mu} \mathcal{B}} \) for (3.5) also in this case. To see that the latter is well defined (although the representation formula (3.4) is not unique), note that each \( \varphi \in w^{-\mu} \mathcal{B} \) can be approximated by a sequence \( (\varphi_k)_{k=1}^\infty \subset \mathcal{C}_c^\infty \) which is bounded in \( w^{-\mu} \mathcal{B} \) and converges in the topology of \( \mathcal{C}_c^\infty \) (simply set \( \varphi_k = \varphi(\cdot) \chi(\cdot/k) \) where \( \chi \in \mathcal{C}_c^\infty \) is a cutoff function supported near 0). From this it is easily seen that the extension to \( \mathcal{B} \) does not depend on the particular choice of representation (3.5). By the same token, it is easy to see that the standard calculation rules for \( \mathcal{D}'_L \), hold when applied to elements of \( \mathcal{B} \), e.g.

\[
\langle f \psi, \varphi \rangle_{w^{-\mu} \mathcal{D}'_L, w^{-\mu} \mathcal{B}} = \langle f, \psi \varphi \rangle_{w^{-\mu} \mathcal{D}'_L, w^{-\mu} \mathcal{B}} \tag{3.6}
\]

for \( f \in w^{-\mu} \mathcal{D}'_L \), \( \psi \in \mathcal{B} \) and \( \varphi \in w^{-\mu} \mathcal{B} \).

As indicated above, \( \mathcal{D}'_L \) is not the topological dual of \( \mathcal{B} \) when \( \mathcal{B} \) is equipped with the standard topology introduced earlier. It is possible to give \( \mathcal{B} \) a different topology such that \( \mathcal{D}'_L \) becomes the dual, but this will not be needed for our purposes. We refer to Schwartz [24, p. 203] for further details. For the convenience of the reader, we end this subsection with a number of known results.

**Proposition 3.2.** Let \( \mu \in \mathbb{R} \). Then \( w^\mu \mathcal{B} \), \( w^\mu \mathcal{F} \) and \( w^\mu \mathcal{D}'_L \) are closed under differentiation, translation and multiplication by functions in \( \mathcal{B} \).

**Proof.** We only prove the statements concerning \( w^\mu \mathcal{B} \), the others being similar or immediate by duality, respectively. It is clear that multiplication is well defined and continuous from \( w^\mu \mathcal{B} \times \mathcal{B} \) to \( w^\mu \mathcal{B} \). Given \( \psi \in w^\mu \mathcal{B} \), we have

\[
d(w^{-\mu} \psi) = (dw^{-\mu}) \psi + w^{-\mu} (d\psi).
\]

The first function is in \( \mathcal{B} \) since this is invariant under differentiation, the second function is in \( \mathcal{B} \) by (3.3), and hence so is the third. In other words, \( \psi' \in w^\mu \mathcal{B} \), as desired. Since \( \mathcal{B} \) is translation invariant, the translation invariance of \( w^\mu \mathcal{B} \) follows by showing that \( w^\mu(\cdot - t)/w^n \in \mathcal{B} \) for each fixed \( t \in \mathbb{R} \). Since

\[
d^m (w^\mu(\cdot - t)) = \sum_{n \leq m} \binom{m}{n} d^n w^\mu(\cdot - t) d^{m-n} w^{-\mu},
\]

this follows immediately by (3.3) and the fact that \( w^\mu(\cdot - t)w^{-\mu} \) is bounded. \( \square \)
Proposition 3.3. Let $\mu \in \mathbb{R}$. Then $\varphi$ in $\mathcal{E}^\infty$ belongs to $w^\mu \mathcal{B}$ if and only if there for each $m \in \mathbb{N}$ is a constant $C_m$ such that $|\varphi^{(m)}(x)| \leq C_m w^\mu(x)$ for all $x \in \mathbb{R}$.

Proof. Suppose the latter holds. Then $d(w^{-\mu} \varphi)$ is easily seen to be in $\mathcal{B}$ by the Leibniz formula and (3.3). Conversely, if $w^{-\mu} \varphi$ is in $\mathcal{B}$, the existence of $C_0 > 0$ such that $|\varphi| < C_0 w^\mu$ is immediate. Corresponding estimates for the derivatives can now easily be obtained by the Leibniz formula, (3.3) and induction.

We finally remark that $\mathcal{S} \subset w^{-\mu} \mathcal{B} \subset \mathcal{E}^\infty$ with continuous inclusions, by which it follows that

$$\mathcal{E}' \subset w^\mu \mathcal{D}'_L \subset \mathcal{S}' .$$  \hspace{1cm} (3.7)

3.2. The $\mathcal{S}'$-convolution

We next recall the definition of the so-called $\mathcal{S}'$-convolution introduced by Hirata and Ogata [16]. We will use the equivalent form given by Shiraishi [25]. For $\varphi \in \mathcal{S}$ we write $\hat{\varphi}(x) = \varphi(-x)$ and extend this operation to $\mathcal{S}'$ by duality.

Definition 3.4. Two tempered distributions $u$ and $v$ in $\mathcal{S}'$ are said to be $\mathcal{S}'$-convolvable if the multiplicative product $u(\hat{v} * \varphi)$ belongs to $\mathcal{D}'_L$, for every $\varphi \in \mathcal{S}$. Then the map from $\mathcal{S}$ to $\mathcal{C}$ given by

$$\varphi \mapsto (u(\hat{v} * \varphi), 1)_{\mathcal{D}'_L, \mathcal{B}}$$

is linear and continuous, and thus defines a tempered distribution denoted by $u * v$.

We remark that when defined, the $\mathcal{S}'$-convolution of $u$ and $v$ is commutative, and satisfies the Fourier exchange formula $(u * v) = \hat{u} \hat{v}$. The notation $u * v$ for the $\mathcal{S}'$-convolution of $u$ and $v$ is justified by the fact that Definition 3.4 coincides with the usual definition of convolution in the sense of distributions whenever the latter definition is applicable.

Proposition 3.5. Let $\mu \in \mathbb{R}$, $f \in w^\mu \mathcal{D}'_L$ and $\psi \in w^{-\mu} \mathcal{B}$ be given, and let $f$ be represented via (3.4). Then the $\mathcal{S}'$-convolution $f * \psi$ exists and equals the function

$$x \mapsto \langle f, \psi(x - \cdot) \rangle_{w^\mu \mathcal{D}'_L, w^{-\mu} \mathcal{B}} = \sum_m \int f_m(t) \psi^{(m)}(x - t) dt. \hspace{1cm} (3.8)$$

Proof. Fix $\varphi \in \mathcal{S}$ and recall Peetre’s inequality which in terms of $w^\mu$ reads

$$w^\mu(t - x) \leq 2^{\frac{|\nu|}{2}} w^\mu(x) w^{|\nu|}(t). \hspace{1cm} (3.9)$$

Since $b := w^\mu \hat{\varphi}$ lies in $\mathcal{B}$ we have that

$$|d^m \hat{\varphi} * \varphi(x)| \leq \int |b(x - t)w^{-\mu}(x - t)\varphi^{(m)}(t)| dt$$

$$\leq \|b\|_\infty 2^{\frac{|\nu|}{2}} w^{-\mu}(x) \int w^{|\nu|}(t)|\varphi^{(m)}(t)| dt \hspace{1cm} (3.10)$$

where the integral is convergent. Thus $\hat{\varphi} * \varphi \in w^{-\mu} \mathcal{B}$ by Proposition 3.3, which by definition implies that $w^\mu \hat{\varphi} * \varphi \in \mathcal{B}$. Since $(\hat{\varphi} * \varphi)f = w^\mu(\hat{\varphi} * \varphi)(w^{-\mu} f)$, where $(w^{-\mu} f) \in \mathcal{D}'_L$, it follows that

$$w^\mu(x - \cdot) \psi(x - \cdot) \in \mathcal{S}' .$$

Therefore, $\hat{\varphi} * \varphi \in \mathcal{S}'$, as desired.
by assumption, we find that $(\tilde{\psi} * \varphi)f \in D'_{L^1}$ as well since $D'_{L^1}$ is closed under multiplication by functions in $\mathcal{B}$. Thus $f$ and $\psi$ are $\mathcal{F}'$-convolvable. By similar considerations, we have that

$$
\langle \psi * f, \varphi \rangle_{\mathcal{F}' \cdot \mathcal{F}} = \langle f(\tilde{\psi} * \varphi), 1 \rangle_{D'_{L^1} \cdot \mathcal{B}} = \langle w^{-\mu} f, w^\mu (\tilde{\psi} * \varphi) \rangle_{D'_{L^1} \cdot \mathcal{B}} = \langle f, \tilde{\psi} * \varphi \rangle_{w^\mu D'_{L^1} \cdot w^{-\mu} \mathcal{B}},
$$

(3.11)

where the second identity is justified by (3.6) since $w^\mu (\tilde{\psi} * \varphi) \in \mathcal{B}$ by the above. By (3.5), we may write $f = \sum d^m f_m$ with $f_m \in L^1(w^{-\mu})$ and let this act on $\psi * \varphi \in w^{-\mu} \mathcal{B}$. Integrate by parts, and note that

$$
(\tilde{\psi} * \varphi)^{(m)}(t) = \tilde{\psi}^{(m)}(t) * \varphi(t) = (-1)^m \int \tilde{\psi}^{(m)}(x-t) \varphi(x) dx
$$

since the convolution $\tilde{\psi} * \varphi$ is well defined in the normal sense. This gives

$$
\langle f, \tilde{\psi} * \varphi \rangle_{w^\mu D'_{L^1} \cdot w^{-\mu} \mathcal{B}} = \int \sum_m f_m(t) \int \psi^{(m)}(x-t) \varphi(x) dx dt.
$$

By Proposition 3.3 and (3.9) we have $|\psi^{(m)}(x-t)| \leq C_m w^{-\mu}(t) w^\mu(x)$. Since $f_m \in L^1(w^{-\mu})$ and $\varphi \in \mathcal{F}$ we may therefore change the order of integration in the last expression above, which gives

$$
\langle f, \tilde{\psi} * \varphi \rangle_{w^\mu D'_{L^1} \cdot w^{-\mu} \mathcal{B}} = \int \sum_m \int f_m(t) \psi^{(m)}(x-t) dt \varphi(x) dx
$$

$$
= \int \langle f, \psi(x) \rangle_{w^\mu D'_{L^1} \cdot w^{-\mu} \mathcal{B}} \varphi(x) dx.
$$

In view of (3.11) we see that $\psi * f$ is given by (3.8), as desired. \qed

**Proposition 3.6.** Let $\mu \in \mathbb{R}$, $f \in w^\mu D'_{L^1}$, and $\psi \in w^{-\mu} \mathcal{B}$ be given. Then $d^n(f * \psi)$ equals $f * \psi^{(n)}$. Moreover, if $\mu > 1$ then $d^n(f * \psi)$ belongs to $L^1(w^{-\mu})$ for all $n \in \mathbb{N}$. In particular, the $\mathcal{F}'$-convolution with $\psi$ preserves $L^1(w^{-\mu})$.

**Proof.** Using the properties of the $\mathcal{F}'$-convolution we have

$$
\langle d^n(\psi * f), \varphi \rangle_{\mathcal{F}' \cdot \mathcal{F}} = (-1)^n \langle \psi * f, \psi^{(n)} \rangle_{\mathcal{F}' \cdot \mathcal{F}} = (-1)^n \langle f(\tilde{\psi} * \varphi), 1 \rangle_{D'_{L^1} \cdot \mathcal{B}}.
$$

Since $\psi^{(n)} \in w^{-\mu} \mathcal{B}$ and $(\psi^{(n)})^* \varphi = (-1)^n \tilde{\psi} * \varphi^{(n)}$, we similarly have

$$
\langle \psi^{(n)} * f, \varphi \rangle_{\mathcal{F}' \cdot \mathcal{F}} = \langle f((\psi^{(n)})^* \varphi), 1 \rangle_{D'_{L^1} \cdot \mathcal{B}} = (-1)^n \langle f(\tilde{\psi} * \varphi), 1 \rangle_{D'_{L^1} \cdot \mathcal{B}},
$$

which completes the proof of the first part.
With this at hand, it is clear that it suffices to prove that \( f \ast \psi \in L^1(w^{-\mu}) \) for fixed \( \psi \in w^{-\mu}\mathcal{B} \). In view of (3.8) we have \[
|f \ast \psi(x)| \leq \sum_m \int |f_m(t)\psi^{(m)}(x-t)|dt
\]
where the sum is finite. By Proposition 3.3 we have \( |\psi^{(m)}(x-t)| \leq C_m w^{-\mu}(x-t) \), so Tonelli’s theorem implies that \[
\|f \ast \psi\|_{L^1(w^{-\mu})} \leq \sum_m C_m \int |f_m(t)|I(t)dt
\]
where \( I(t) = \int w^{-\mu}(x-t)w^{-\mu}(x)dx \). Noting that \( w(x) \sim 1 + |x| \), an application of Alvarez et al. [2, Lemma 2.8] shows that \( 0 < I(t) \leq C_\mu w^{-\mu}(t) \) for some constant \( C_\mu \) (this is where the assumption \( \mu > 1 \) is used). Since \( f_m \in L^1(w^{-\mu}) \) for each \( m \), the integrals in (3.12) are therefore convergent, and we are done. \( \square \)

3.3. The \( S' \)-convolution and \( \mathcal{P}_{\alpha,y} \)

We now turn to the problem of finding the optimal class of tempered distributions that are \( S' \)-convolvable with the kernel \( \mathcal{P}_{\alpha,y} \). Note that with \( w \) as in Definition 3.1, we have
\[
w^{\alpha+2}(x) = \frac{(1 + x^2)^{(\alpha+2)/2}}{2},
\]
which means that \( w^{-\alpha-2} \) is modulo a scaling factor equal to \( |\mathcal{P}_{\alpha,1}| \). Also note that
\[
\mathcal{P}_{\alpha,y}(\eta) = \mathcal{P}_{\alpha,y}(-\eta) = \mathcal{P}_{\alpha,y}(\eta),
\]
where the last identity is easily checked using Definition 2.1. We will also need the following estimates of \( \mathcal{P}_{\alpha,y} \).

Lemma 3.7. Let \( \alpha > -1 \). Then \( \mathcal{P}_{\alpha,y} \in w^{-\alpha-2}\mathcal{B} \). More precisely, for all \( m,n \in \mathbb{N} \) and \( t \in \mathbb{R} \) we have
\[
|\partial_x^m \partial_y^n \mathcal{P}_{\alpha,y}(x-t)| \leq C_{\alpha,m,n}(y)w^{-\alpha-2-m}(x)w^{\alpha+2+m}(t)
\]
where \( C_{\alpha,m,n}(y) \) is a constant that depends continuously on \( y > 0 \).

Proof. We first prove (3.14) for \( t = 0 \). Recall that \( \mathcal{P}_{\alpha,y}(x) \) is a constant multiple of \( y^{\alpha+1}(x-iy)^{-1}(x+iy)^{-(\alpha+1)} \). Using the Leibniz formula, it is straightforward to check that \( \partial_x^m \) applied to this function is a linear combination of terms of the form
\[
\frac{y^{\alpha+1}}{(x-iy)^{1+j}(x+iy)^{\alpha+1+m-j}},
\]
where \( j = 0,1,\ldots,m \). Applying \( \partial_y^n \) to this yields a linear combination of terms of the form
\[
\frac{y^{\alpha+1-k}}{(x-iy)^{1+j}(x+iy)^{\alpha+1+l}},
\]
where \( j + \ell = m + n - k \) and \( 0 \leq k \leq n \). Each of these terms is clearly bounded by \( y^\beta w^{-\alpha - 2 - m}(x/y) \) where \( \beta = -1 - m - n \). Noting that
\[
w^{-\alpha - 2 - m}(x/y) \leq w^{-\alpha - 2 - m}(x) \max(1, y^{\alpha + 2 + m}),
\]
we thus conclude that
\[
|\partial_x^n \partial_y^m \mathcal{P}_{\alpha,y}(x)| \leq C_{\alpha,m,n}(y)w^{-\alpha - 2 - m}(x)
\]  
(3.15)
for some function \( C_{\alpha,m,n} \) continuous in \( y > 0 \). This is (3.14) for \( t = 0 \). The general case follows by replacing \( x \) with \( x - t \) in (3.15) and applying Peetre’s inequality (3.9) to \( w^{-\alpha - 2 - m}(x - t) \). Finally, that \( \mathcal{P}_{\alpha,y} \in w^{\alpha + 2} \mathcal{B} \) is an immediate consequence of (3.14) and Proposition 3.3.

The next result is the main theorem of this section. Similar results have been obtained when \( \mathcal{P}_{\alpha,y} \) is replaced by \( |\mathcal{P}_{\alpha,y}| \) (allowing also for \( x \in \mathbb{R}^n \)), see Alvarez et al. [3, Theorem 10] for the case \( \alpha = 0 \), and Wittsten [28, Theorem 3.3] for general \( \alpha > 0 \).

**Theorem 3.8.** Let \( \alpha > -1 \). Let \( f \in \mathcal{S}' \). Then the following assertions are equivalent:

(i) \( f \in w^{\alpha + 2} \mathcal{G}_{L^1}' \).

(ii) \( f \) is \( \mathcal{S}' \)-convolvable with \( \mathcal{P}_{\alpha,y} \) for each \( y > 0 \).

**Proof.** Assume first that (i) holds. Lemma 3.7 implies that \( \mathcal{P}_{\alpha,y} \in w^{-\alpha} \mathcal{B} \), which by Proposition 3.5 implies that \( \mathcal{P}_{\alpha,y} \) is \( \mathcal{S}' \)-convolvable with \( f \).

Assume next that (ii) holds, and introduce a cutoff function \( \chi \in \mathcal{C}_c^\infty \) taking values in \([0,1]\) such that \( \chi(x) \) is identically equal to 1 for \( |x| \leq 1/2 \), positive for \( |x| < 1 \) and vanishes for \( |x| \geq 1 \). Set \( \psi_k = \chi(k/\ell) \), write \( f = \psi_k f + (1 - \psi_k) f \) and note that \( f_1 = \psi_k f \in \mathcal{S}' \subset w^{\alpha + 2} \mathcal{G}_{L^1}' \) by (3.7), so it remains to prove that \( (1 - \psi_k) f \in w^{\alpha + 2} \mathcal{G}_{L^1}' \) if \( k \) is sufficiently large.

Next, recall (3.13) and consider the convolution
\[
\mathcal{P}_{\alpha,y} \ast \chi(x) = \frac{2^\alpha e^{-i \alpha \pi / 2}}{\pi} \int_{|t| < 1} \frac{y^{\alpha + 1}(z - t)^\alpha}{|z - t|^{2(\alpha + 1)}} \chi(t) dt, \quad z = x + iy.
\]
For fixed \( \alpha \) and \( y \), geometrical considerations show that if \( |x| \) is sufficiently large, then \( t \mapsto \Re(z - t)^\alpha \) and \( t \mapsto \Im(z - t)^\alpha \) do not change sign when \( |t| < 1 \). Moreover, we either have \( |\Re(z - t)^\alpha| \geq \frac{\alpha}{2} |z - t|^\alpha \) or \( |\Im(z - t)^\alpha| \geq \frac{\alpha}{2} |z - t|^\alpha \) for \( |t| < 1 \). It follows that
\[
|\mathcal{P}_{\alpha,y} \ast \chi(x)| \geq C_\alpha \int_{|t| < 1} \frac{y^{\alpha + 1}}{|2z - t|^{\alpha + 2}} |\chi(t)| dt, \quad z = x + iy,
\]
if \( |x| \) is sufficiently large. We now fix \( k \geq 2 \) such that this is the case for all \( x \in \text{supp}(1 - \psi_k) \). Then \( |x - t| \leq 2|x| \) for \( |t| < 1 \) and \( x \in \text{supp}(1 - \psi_k) \) since \( k \geq 2 \).

Combining these facts, a short calculation yields
\[
|\mathcal{P}_{\alpha,y} \ast \chi(x)| \geq \tilde{C}_\alpha \frac{y^{\alpha + 1}}{(x^2 + y^2)^{(\alpha + 2)/2}} \|\chi\|_{L^1}, \quad x \in \text{supp}(1 - \psi_k).
\]  
(3.16)
We now claim that

\[(1 - \psi_k)(\mathcal{P}_{\alpha,y} * \chi)^{-1} \in w^{\alpha+2}\mathcal{B} \quad \text{for each } y > 0. \quad (3.17)\]

Indeed, \(\mathcal{P}_{\alpha,y} \in w^{-\alpha-2}\mathcal{B}\) by Lemma 3.7, so by (3.10) we have \(\mathcal{P}_{\alpha,y} * \chi \in w^{-\alpha-2}\mathcal{B}\). Hence, derivatives of \(\mathcal{P}_{\alpha,y} * \chi\) are bounded by a constant multiple of \(w^{-\alpha-2}\) (Proposition 3.3). Using also (3.16) together with the Leibniz and the Faà di Bruno formulas we find that derivatives of \((1 - \psi_k)(\mathcal{P}_{\alpha,y} * \chi)^{-1}\) can be estimated by a constant multiple of \(w^{\alpha+2}\). Hence, (3.17) follows by virtue of Proposition 3.3. Finally, assumption (ii) implies that we have \((\mathcal{P}_{\alpha,y} * \chi)f \in \mathcal{D}'_{L^1}\) by virtue of Definition 3.4, which gives

\[(1 - \psi_k)(x)f(x) = w^{\alpha+2}(x) \cdot \frac{1 - \psi_k(x)}{w^{\alpha+2}(x)\mathcal{P}_{\alpha,y} * \chi(x)} \cdot \mathcal{P}_{\alpha,y} * \chi(x)f(x).\]

Since \(\mathcal{D}'_{L^1}\) is closed under multiplication by functions in \(\mathcal{B}\), the right-hand side thus defines an element in \(w^{\alpha+2}\mathcal{D}'_{L^1}\), which completes the proof. \(\square\)

**Definition 3.9.** Let \(\alpha > -1\). Let \(f \in w^{\alpha+2}\mathcal{D}'_{L^1}\). The Poisson integral of \(f\) with respect to the kernel \(\mathcal{P}_{\alpha}\) is defined as the function

\[\mathcal{P}_{\alpha}[f] : z \mapsto \mathcal{P}_{\alpha,y} * f(x), \quad z = x + iy \in \mathbb{C}_+.\]

Henceforth, we will write \(\mathcal{P}_{\alpha}[f]\) only when referring to the above function on \(\mathbb{C}_+\); the value of \(\mathcal{P}_{\alpha}[f]\) at \(z = x + iy\) will usually still be written as \(\mathcal{P}_{\alpha,y} * f(x)\), and we will continue to write \(\mathcal{P}_{\alpha,y} * f\) when discussing the map \(x \mapsto \mathcal{P}_{\alpha}[f](x + iy)\).

**Proposition 3.10.** Let \(\alpha > -1\) and \(f \in w^{\alpha+2}\mathcal{D}'_{L^1}\). Then

\[\partial^k_y \partial^\ell_x (\mathcal{P}_{\alpha,y} * f(x)) = (\partial^k_y (\mathcal{P}_{\alpha,y})^{(\ell)}) * f(x)\]

for all \(k, \ell \in \mathbb{N}\). In particular, \(\mathcal{P}_{\alpha}[f]\) is \(\alpha\)-harmonic.

**Proof.** Suppose that we have shown (3.18) for \(\ell = 0\). By Lemma 3.7 and Proposition 3.3 we have that \(\partial^\ell_y \mathcal{P}_{\alpha,y} \in w^{-\alpha-2}\mathcal{B}\), so the general form of (3.18) then follows by Proposition 3.6. We thus fix \(\ell = 0\). Proposition 3.5 then gives

\[(\partial^k_y \mathcal{P}_{\alpha,y}) * f(x) = \sum_m (-1)^m \int f_m(t) \partial^m_y \mathcal{P}_{\alpha,y}(x - t) \, dt.\]

Using the concrete estimate for \(\partial^k_y \mathcal{P}_{\alpha,y}\) in Lemma 3.7, we see that for fixed \(x\) and \(y\) in some compact subinterval of \((0, \infty)\), each integrand is dominated by a constant times the integrable function \(f_m w^{-\alpha-2}\). By standard integration theory (see e.g. Folland [13, Theorem 2.27]) it follows that \(\partial^k_y\) can be moved outside of the integral, and hence (3.18) holds. \(\square\)

4. **The \(\alpha\)-harmonic Dirichlet problem**

4.1. **Existence of solutions**

Let \(\alpha > -1\) and \(f \in w^{\alpha+2}\mathcal{D}'_{L^1}\). Since \(\mathcal{P}_{\alpha,y} \in w^{-\alpha-2}\mathcal{B}\) by Lemma 3.7, Proposition 3.6 ensures that \(\mathcal{P}_{\alpha,y} * f\) belongs to \(L^1(w^{-\alpha-2})\) for all \(y > 0\). As \(L^1(w^{-\alpha-2})\) is a subspace of \(w^{\alpha+2}\mathcal{D}'_{L^1}\) in the usual sense, we may therefore consider the convergence of \(\mathcal{P}_{\alpha,y} * f\) in \(w^{\alpha+2}\mathcal{D}'_{L^1}\) as \(y \to 0\).
Theorem 4.1. Let \( \alpha > -1 \). Let \( f \in w^{\alpha+2}D^1 \) and set \( u = \mathcal{P}_\alpha[f] \). Then \( u_y \to f \) in \( w^{\alpha+2}D^1 \) as \( y \to 0 \), where \( u_y(x) = u(x + iy) \) for \( y > 0 \).

Proof. We will essentially adapt the proof of Alvarez et al. [2, Theorem 3.6]. Suppose first that we have already proved that if \( g \in L^1(w^{-\alpha-2}) \) then \( \mathcal{P}_{\alpha,y} * g \to g \) in \( L^1(w^{-\alpha-2}) \) as \( y \to 0 \). Since \( L^1(w^{-\alpha-2}) \) is continuously embedded in \( w^{\alpha+2}D^1 \), it follows that \( \mathcal{P}_{\alpha,y} * g \to g \) in \( w^{\alpha+2}D^1 \) as \( y \to 0 \). Now let \( f \in w^{\alpha+2}D^1 \). By (3.4) we can then write \( f \) as a finite sum with terms of the form \( d^m f_m \) with \( f_m \in L^1(w^{-\alpha-2}) \), and in view of (3.8) we have

\[
\mathcal{P}_{\alpha,y} * f = \sum_m (\mathcal{P}_{\alpha,y})^{(m)} * f_m.
\]

By Proposition 3.10 we have \( (\mathcal{P}_{\alpha,y})^{(m)} * f_m(x) = \partial^m_x (\mathcal{P}_{\alpha,y} * f_m)(x) \). Since the operation of differentiation is continuous in \( w^{\alpha+2}D^1 \), by Proposition 3.2, this gives

\[
\mathcal{P}_{\alpha,y} * f = \sum_m (\mathcal{P}_{\alpha,y})^{(m)} * f_m = \sum_m \partial^m_x (\mathcal{P}_{\alpha,y} * f_m) \to \sum_m d^m f_m = f
\]

in \( w^{\alpha+2}D^1 \) as \( y \to 0 \). Hence it suffices to prove that if \( f \in L^1(w^{-\alpha-2}) \) then \( \mathcal{P}_{\alpha,y} * f \to f \) in \( L^1(w^{-\alpha-2}) \) as \( y \to 0 \).

Suppose therefore that \( f \in L^1(w^{-\alpha-2}) \). Since \( \alpha > -1 \), the definition of \( w^{\alpha+2} \) ensures that \( w^{-\alpha-2}(x)dx \) is a finite, complete, regular measure on \( \mathbb{R} \), which implies that the compactly supported continuous functions \( \mathcal{C}_c \) are dense in \( L^1(w^{-\alpha-2}) \). Given \( \varepsilon > 0 \) we let \( g \in \mathcal{C}_c \) satisfy

\[
\| f - g \|_{L^1(w^{-\alpha-2})} < \varepsilon.
\]

Next, note that since \( 0 < w^{-\alpha-2}(x) \leq 1 \) for \( x \in \mathbb{R} \) we get from (2.1)–(2.2) that

\[
\| \mathcal{P}_{\alpha,y} * g - g \|_{L^1(w^{-\alpha-2})} \leq \int [\| \mathcal{P}_{\alpha,y}(x) \| |g(t - x) - g(t)|]dxdt = \int [\| \mathcal{P}_{\alpha,y}(x) \| \| \tau_x g - g \| _{L^1}]dx,
\]

where \( \tau_x \) denotes translation by \( x \) and the last identity follows by Tonelli’s theorem. It is well known that \( |\tau_x g - g|_{L^1} \to 0 \) as \( x \to 0 \), see Bochner [7, Theorem 1.2.1]. Since \( \mathcal{P}_{\alpha,y} \) enjoys the usual properties of kernel functions it follows that

\[
\| \mathcal{P}_{\alpha,y} * g - g \|_{L^1(w^{-\alpha-2})} \to 0 \quad \text{as} \quad y \to 0,
\]

see Bochner [7, Theorem 1.3.2]. Now,

\[
\| \mathcal{P}_{\alpha,y} * f - f \|_{L^1(w^{-\alpha-2})} \leq \| \mathcal{P}_{\alpha,y} * (f - g) \|_{L^1(w^{-\alpha-2})} + \| \mathcal{P}_{\alpha,y} * g - g \|_{L^1(w^{-\alpha-2})} + \| g - f \|_{L^1(w^{-\alpha-2})},
\]

so it only remains to estimate the first term in the right-hand side. Note that

\[
\| \mathcal{P}_{\alpha,y} * (f - g) \|_{L^1(w^{-\alpha-2})} \leq \int (|\mathcal{P}_{\alpha,y}| * w^{-\alpha-2})(t)|f(t) - g(t)|dt
\]

by Tonelli’s theorem and the fact that \(|\mathcal{P}_{\alpha,y}| \) is an even function. One can verify that

\[
0 < |\mathcal{P}_{\alpha,y}| * w^{-\alpha-2}(t) \leq C_\alpha w^{-\alpha-2}(t)
\]

for some constant \( C_\alpha \) independent of \( y \), and we
refer to the proof of Wittsten [28, Theorem 4.3] for details. By virtue of (4.3) we find that 
\[ \| \mathcal{P}_{\alpha,y} * (f - g) \|_{L^1(w^{-\alpha-2})} \leq C_\alpha \| f - g \|_{L^1(w^{-\alpha-2})}. \]
In view of (4.1)–(4.2) this implies that 
\[ \| \mathcal{P}_{\alpha,y} * f - f \|_{L^1(w^{-\alpha-2})} \leq (C_\alpha + 2)\epsilon \]
for any sufficiently small \( \epsilon > 0 \), which completes the proof.

We now state the main result of this section.

**Corollary 4.2.** Let \( \alpha > -1 \). Let \( f \in w^{\alpha+2}G_{L^1}^\prime \). Then \( \mathcal{P}_\alpha[f] \) is a solution to the Dirichlet problem (0.3).

**Proof.** Set \( u = \mathcal{P}_\alpha[f] \) and note that \( u \) is \( \alpha \)-harmonic by Proposition 3.10. Moreover, \( w^{\alpha+2}G_{L^1}^\prime \subset \mathcal{P} \) with continuous inclusion, so Theorem 4.1 implies that \( u_y \to f \) in \( \mathcal{P} \) as \( y \to 0 \), where \( u_y(x) = u(x + iy) \) for \( y > 0 \) in accordance with (0.4).

**Remark.** When \( \alpha > -1 \) we have \( L^\infty \subset w^{\alpha+2}G_{L^1}^\prime \), since \( w^{-\alpha-2} \) is then an integrable function. Indeed, each \( f \in L^\infty \) defines an element in \( w^{\alpha+2}G_{L^1}^\prime \) through integration: for \( \varphi \in w^{-\alpha-2}G\) we get

\[
|\langle f, \varphi \rangle_{w^{\alpha+2}G_{L^1}^\prime, w^{-\alpha-2}G}| \leq \int |f(t)\varphi(t)|dt \leq C_\alpha \| f \|_{L^\infty} \sup_t |w^{\alpha+2}(t)\varphi(t)|,
\]
where \( C_\alpha = \int w^{-\alpha-2}(t)dt < \infty \), and \( \sup |w^{\alpha+2}\varphi| \) is a semi-norm in \( w^{-\alpha-2}G \). Hence, the remarks following Proposition 2.3 about the convolution \( \mathcal{P}_{\alpha,y} * f \) solving (0.3) when \( f \in L^\infty \) follow from the general theory developed for boundary values in \( w^{\alpha+2}G_{L^1}^\prime \).

### 4.2. Uniqueness revisited

We first investigate asymptotic growth behavior of the Poisson integral \( \mathcal{P}_\alpha[f] \) when \( f \in w^{\alpha+2}G_{L^1}^\prime \). We shall obtain growth estimates comparable to those satisfied by the classical \( (\alpha = 0) \) Poisson integral of \( p \)-summable functions proved by Siegel and Talvila [26]. For convenience, we recall (a slight reformulation of) Wittsten [28, Lemma 5.1]. We shall write \( S_r \) to denote the set

\[
S_r = \mathbb{C}_+ \cap \{ x + iy \in \mathbb{C} : x^2 + y^2 = r^2 \}, \quad r > 0.
\]

For each \( k \in \mathbb{N} \), we also define \( L_{\alpha,k} : \mathbb{C}_+ \to \mathbb{R} \) by

\[
L_{\alpha,k}(x + iy) = \frac{y^{\alpha+1}}{(x^2 + y^2)^{(\alpha+2+k)/2}}.
\]

(Thus, modulo a scaling factor we have \( |\mathcal{P}_\alpha| = L_{\alpha,0} \).)

**Lemma 4.3.** Let \( \alpha > -1 \). If \( \mu \leq \alpha + 2 \) and \( f \in L^1(w^{-\mu}) \) then

\[
\int L_{\alpha,k}(x - t + iy)|f(t)|dt \leq C_\mu I(r)t^\mu/y^{1+k}, \quad x + iy \in S_r,
\]
where \( I(r) \to 0 \) as \( r \to \infty \).

Note that by the proof of Wittsten [28, Lemma 5.1] we in fact have

\[
I(r) \equiv I_\mu(r) = \int \frac{|f(t)|}{(t^2 + r^2)^{(\alpha+2)/2}}dt, \quad r > 0.
\]
Theorem 4.4. Let $\alpha > -1$. Let $f \in w^{\alpha+2}P_{L^1}$. Then there exists an $m \in \mathbb{N}$ depending only on the distribution $f$ such that the Poisson integral $P_{\alpha}[f]$ satisfies

$$
\sup_{S_r} |(y/r)^{m+1}P_{\alpha,y} * f(x)| = o(r^{\alpha+1}) \quad \text{as } r \to \infty. \quad (4.4)
$$

In particular, $P_{\alpha}[f]$ is of polynomial growth in $iy_0 + C$ for all $y_0 > 0$.

Proof. By the representation formula (3.4) we have $f = \sum_{k=0}^{m} d^k f_k$ for some $m \in \mathbb{N}$, where $f_k \in L^1(w^{-\alpha-2})$. As before (see (3.8)) we then have

$$
P_{\alpha,y} * f(x) = \sum_{k=0}^{m} \int f_k(t)(P_{\alpha,y})^{(k)}(x-t)dt.
$$

Note that

$$
|(P_{\alpha,y})^{(k)}(x-t)| \leq C_{\alpha,k} \frac{y^{\alpha+1}}{(|x-t|^2 + y^2)^{(\alpha+2+k)/2}} = C_{\alpha,k} I_{\alpha,k}(x-t + iy).
$$

Applying Lemma 4.3 with $\mu = \alpha + 2$ gives the estimate

$$
y^{m+1}|P_{\alpha,y} * f(x)| \leq \sum_{k=0}^{m} C_k I_k(r)^{\alpha+2}y^{m-k}. \quad (4.5)
$$

For $r > 1$ we have $y^{m-k} \leq r^{m-k} \leq r^m$, which inserted into (4.5) immediately gives (4.4) since $I_k(r) \to 0$ as $r \to \infty$. That $x + iy \mapsto P_{\alpha,y} * f(x)$ has polynomial growth in each region $y \geq y_0 > 0$ is also an easy consequence of (4.5). \qed

The following two theorems constitute the main conclusions of this paper.

Theorem 4.5. Let $\alpha > -1$. Let $u$ be $\alpha$-harmonic in $C_+$, and assume that

(i) $z \mapsto u(iy_0 + z)$ has polynomial growth for all $y_0 > 0$,

(ii) $u(\cdot + iy)$ converges to $f$ in $\mathcal{F}$ as $y \to 0$, where $f \in w^{\alpha+2}P_{L^1}$.

Then $u = P_{\alpha}[f] + u_0$ in $C_+$, where $u_0$ is an $\alpha$-harmonic quasi-polynomial.

Proof. By Theorem 4.4 it follows that $P_{\alpha}[f]$ satisfies conditions (i). In view of (ii) and Corollary 4.2 we can apply Theorem 1.7 and Proposition 1.9 to $u - P_{\alpha}[f]$, by which the desired result follows. \qed

Theorem 4.6. Let $\alpha > -1$. Let $u$ be $\alpha$-harmonic in $C_+$, and assume that

(i) $z \mapsto u(iy_0 + z)$ has polynomial growth for all $y_0 > 0$,

(ii) $u(\cdot + iy)$ converges to $f$ in $\mathcal{F}$ as $y \to 0$, where $f \in w^{\alpha+2}P_{L^1}$, and

(iii) there is a $\delta > 0$ such that $\lim_{y \to \infty} y^{-\alpha-1}|u(x + iy)| = 0$ for all $|x| < \delta$.

Then $u = P_{\alpha}[f]$ in $C_+$.

Proof. Theorem 4.4 shows that $P_{\alpha}[f]$ also satisfies condition (iii), so the conclusion follows by applying Corollary 1.10 to $u - P_{\alpha}[f]$. \qed
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