Remarks on octonion modules I
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1 Introduction

Here I extend previous work about quaternion modules [2] to the case of octonions.

For a while, we let A be any finite dimensional algebra over the reals $\mathbb{R}$.[1] Only later, we shall specialize to the ring $\mathbb{O}$ of *octonions* (or *octaves* or *Cayley-Graves numbers*), which is an 8-dimensional algebra over $\mathbb{R}$.

1.1 Commutator and associator

Let A be finite dimensional algebra over the reals $\mathbb{R}$. A is said to be *commutative* if $\forall\, x, y\; xy = yx$. The it is customary to introduce for $x, y \in A$ the quantity $\mathfrak{C}(x, y) =: [x, y] =: xy - yx$, called the *commutator*.

Similarly, one says that A is *associative* if $\forall\, x, y, z\; x(yz) = y(xz)$. In this case one introduces, similarly or $x, y \in A$ the quantity $\mathfrak{A}(x, y, z) =: [x, y, z] =: x(yz) - y(xz)$, called the *associator*.

1.2 Meurman's Lemma

In this section we take $A = \mathbb{O}$. On my request Arne Meurman proved the following result.

Lemma 1 (Meurman's Lemma) *Let $a \in \mathbb{O}$. Assume that $\forall x, y \in \mathbb{O}$ holds $[x, y, a] = 0$. Then $a \in \mathbb{R}$.*

Indeed, Meurman outlined two independent proofs of Lemma 1. The first is based on the fact that the automorphism group of $\mathbb{O}$ is isomorphic to a compact Lie group of type G_2. In the second one, one first expresses the condition in terms of the real components of the octonions. This gives a

[1] Algebras can be defined over any (commutative) field K [4], but, being basically still an analysist, only real scalars matter for me.

system of homogeneous linear equations in the variable a, in number $7 \times 7 \times 8$. So everything blows down to showing that the only solution, up to a multiplicative constant, is $(1,1,1,1,1,1,1,1)$ (a vector in $\mathbb{R}^8$ with all 8 entries equal to unity).

Remark 1 *There is a similar lemma in the case of quaternions* $\mathbb{H}$*, involving the* commutator*, [2], Lemma 1:* Let $x \in \mathbb{H}$. If $\forall x \in \mathbb{H}\, [x, y] = 0$, then $y \in R$. *It is much easier to prove.*

2 Basic definitions

Let us return to the case of a general algebra A.

Definition 1 *By a* left A-module M *we mean an A-space eqipped with an operation* $A \times M \to M : (\lambda, x) \mapsto \lambda x$ *which is $\mathbb{R}$-linear in the second argument:*

$$\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2; \quad \lambda(cx) = c\lambda x \quad (c \in A).$$

Remark 2 *In the same way one defines* right A-module*. Then one has an operation* $A \times M \to M : (\lambda, x) \to x\lambda$*. Thus, on this level, left and right modules can are undistinguishable.*

Remark 3 *It might be advantageous to use, sometimes, an alternative "functional" notation, writing instead* $f(\lambda, x) =: \lambda x$, $g(x, \lambda) =: x\lambda$.

Definition 2 *A* free finitely generated left $\mathbb{O}$-module *is a left module M such that there exists in M a basis $e_1, \ldots, e_n$ such that that each element $x \in M$ has a unique representation $x = x^1 e_1 + \ldots + x^n e_n$ or, more compactly, $x = \sum_k x^k e_k$ $(x^k \in A)$. Moreover, it is assumed that*

$$\forall\, \lambda \in A\ \lambda x = \sum_k (\lambda x^k) e_k. \tag{1}$$

Such a basis will be called a free basis*. The number n will be called the* dimension *of M. We speak also of an n-dimensional free left A-module.*

In other words, M can be identified with the Cartesian product via the map $M \to A^n : x \mapsto (x^1, \ldots, x^n)$. The standard left A-module structure on A^n is multiplication of the components by elements of A from the left: $\lambda(x^1, \ldots, x^n) =: (\lambda x^1, \ldots, \lambda x^n)$.

3 Two free bases

Let M be a free n-dimensional left A-module, and consider any two free bases $e_1, \ldots, e_n$ and $g_1, \ldots, g_n$ in M. Then every element $x \in M$ can be written as

$$x = x^1 e_1 + \ldots + x^n e_n = \sum_k x^k e_k, \tag{2}$$

with suitable elements $x^k \in A$, and also as

$$x = x^1 g_1 + \ldots + x^n g_n = \sum_\ell y^\ell g_\ell, \tag{3}$$

with some other elements $y^\ell \in A$.

Let us investigate the relation between these two bases. As the g_ℓ are all elements of M, we can express them in terms of the first basis $e_1, \ldots, e_n$:

$$g_\ell = \sum_k a_\ell^k e_k. \tag{4}$$

Thus there evolves a *matrix* (a_ℓ^k), called the *transition matrix.*

We get then from (3, 4)

$$x = \sum_\ell y^\ell (\sum_k a_\ell^k e_k) = \sum_k \sum_\ell (y^\ell a_\ell^k) e_k.$$

Now, avoking the fact that the e_k make up a basis, this again gives

$$x = \sum_k (\sum_\ell y^\ell a_\ell^k) e_k. \tag{5}$$

Comparing (5) with (2) gives

$$x^k = \sum_\ell y^\ell a_\ell^k. \tag{6}$$

Thus, we have found the *formula for change of coordinates.*

Let λ be an arbitrary element in A. Then (1) holds true, and we find, using also (6),

$$\lambda x = \sum_k (\lambda x^k) e_k = \sum_k \underline{(\lambda(\sum_\ell y^\ell a_\ell^k))} e_k$$

On the other hand, using (5), the fact that the g_ℓ make a free basis and also invoking (4), one obtains

$$\lambda x = \lambda(\sum y^\ell g_l) = \sum_{\ell,k} (\lambda(y^\ell)) g_\ell = \sum_\ell (\lambda(y^\ell)) \sum_k a_\ell^k e_k = \sum_{k,\ell} \underline{(\lambda(y^\ell)) a_\ell^k} e_k. \tag{7}$$

Finally, comparing the coefficients of e_k in the two above *underlined* expressions for λx obtained, we see that

$$\forall k, \ell \ \sum_\ell \lambda(y^\ell a_\ell^k) = (\lambda(y^\ell))a_\ell^k. \tag{8}$$

To proceed further, we observe that in (8) the variables λ and y^ℓ can be chosen *ad libitum*. Therefore we use the following *trick*. We keep λ fixed but put $y^\ell = t^\ell y$, where t is a real variable and y some fixed element in A. This gives

$$\sum_\ell t^\ell((\lambda y)a_\ell^k) = \sum_\ell t^\ell(\lambda y)a_\ell^k.$$

or even

$$\sum_\ell t^\ell \left(((\lambda y)a_\ell^k) - (\lambda y)a_\ell^k\right) = \sum_\ell t^\ell[\lambda, y, a_\ell^k].$$

Next, we envoke the following elementary and probably well-known result.

Lemma 2 *Let u_ℓ be finitely many elements in some vector space V over $\mathbb{R}$. Assume that $\forall t \sum_\ell t^\ell u_\ell = 0$. Then $\forall \ell\, u_\ell = 0$.*

Proof. By a well-known, elementary argument. Let us put $t = 0$. Then it follows that $u_0 = 0$ and so $\forall t \sum_{\ell>0} t^\ell u_\ell = 0$. Thus one has reduced the number of non-zero coefficients by one. Proceeding by induction we see that all coefficients vanish.QED

At is moment we specialize *to the case octonions, $A = \mathbb{O}$.* This means that we are free to use Lemma 1. Thus we arrive at the following result

Theorem 1 *We have $\forall\, k, \ell \ \ a_\ell^k \in \mathbb{R}$.*

In other words, the transition matrices $a =: (a_\ell^k)$, in this special case, are all *real*, We shall soon see that they indeed form a group, isomorphic to the real general group $\mathrm{GL}(n, \mathbb{R})$.

First we prove an easy result.

Theorem 2 *The above matrix $a =: (a_\ell^k)$ is non-singular.*

Proof Let us assume the contrary. Then there exists a non-zero vector $y = (y^\ell)$ such that

$$\forall k \sum_\ell a_\ell^k y^\ell = 0.$$

Let $x = \sum_\ell y^\ell g_\ell$. As $y \neq 0$ and the g_ℓ make up a basis, then also $x \neq 0$. But from the above linear equations, we see that all $x^k = 0$, so $x = 0$. Contradiction. QED

Next, we establish the following.

Theorem 3 *Assume that all* $a_\ell^k \in \mathbb{R}$*. Let the* g_ℓ *be* defined *by* $g_\ell = a_\ell^k \varepsilon_k$*. Then the elements* g_ℓ *give a free basis.*

Proof. By Theorem 2, the matrix $a =: a_\ell^k$ has an inverse $b = a^{-1} =: b_\ell^k$, which is a real matrix. In matrix notation $ab = \mathbf{1}$, $ba = \mathbf{1}$, where $\mathbf{1}$ is the n-dimensional unit matrix. But with real matrices one can operate as in ordinary commutative Linear Algebra, even if the vectors involved are octonian. So rewriting (6) as $x = ya$, where x and y represent elements pf our $\mathbb{O}$-module M. We obtain now $y = xa^{-1} = xb$. This clearly show that the g_ℓ yield a free basis. QED

All that we need to prove that transition matrices under view form a group is to show that the product of any two such matrices is again a transition matrix. This we do in the next section.

4 Three bases, or more

In this Section we let A again be an arbitrary $\mathbb{R}$-algebra and, as before, M a n-dimensional A-module. Thus, let there be given three bases in M, $e_1, \ldots, e_n$, $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$. Then we have the relations (as before) $e_k = a_k^\ell g_\ell$ and (new) $g_\ell = b_\ell^m h_m$. They correspond to the two formulae for change of coordinates

$$x^k = \sum_\ell y^\ell a_\ell^k, \tag{9}$$

respectively

$$y^\ell = \sum_m z^m b_m^\ell. \tag{10}$$

If we compose these two transitions, we find

$$e_k = \sum_k a_k^\ell (\sum_m b_\ell^m h_m) = \sum_\ell (\sum_m a_k^\ell b_\ell^m) h_m. \tag{11}$$

Let us set

$$c_k^m =: \sum_\ell a_k^\ell b_\ell^m, \tag{12}$$

Then we may also write (11) as

$$x^k = \sum_\ell z^m c_m^k. \tag{13}$$

Let now $A = \mathbb{O}$ be the ring of octonions. Then it follows from (12) that the matrix (c_k^m) is the product of the two matrices (a_k^ℓ) and (b_ℓ^m).

Thus we have proved.

Theorem 4 *Let $A = \mathbb{O}$. Then the product of any two transition matrices is again a transition matrix, and so an element of $GL(n, \mathbb{R})$.*

The above formulae remind suspiciously about the formulae for multiplying matrices and composing linear transformations – in the associative case. But we are in the nonassociative case. So we must proceed with utmost care. *Otherwise, we shall fall into the associative trap!* This we do in next section.

5 Monoids

Usually, monoids are taken associative. Here, we shall consider them quite generally, dropping such an assumption. Thus we define as follows R.

Definition 3 *By a* monoid *we intend a set G equipped with a binary operation $G \to G \times G : (x, y) \mapsto x \cdot y$ (or, simply, xy).*

We say that G is *unary* if there exist an element u, a (bilateral) unit, such that $\forall x\, ux = xu = x$. (Left and right units are defined in a similar awy.

Remark 4 *As in Remark 3, it is sometimes be expedient to use a functional notation, writing $h : G \to G \times G$ for the function $h(x, y) = x \cdot y$ (or just xy).*

After these esoteric preparations, let us return to our previous situation, A an $\mathbb{R}$-algebra, M an n-dimensional A-module. Then we let $\mathfrak{M}_n(A)$ be the set of all $n \times n$ matrices (a^k_ℓ). It becomes a monoid (in our above sense), if we take (12) as the multiplication. Clearly, there is a unit, the matrix $\mathbf{1} = \mathbf{1}_n = (\delta^k_\ell)$, with $\delta^k_\ell = 1$ if $k = \ell$, 0 otherwise.

6 Preview

In Part II are to use the previous methoda applied to Wachs modules also.

Recall that in the case of *quaternions* $\mathbb{H}$ the spectral theory of self-adjoint and normal operators was developed by Oswald Teichmüller [3] in his 1935 Göttingen thesis. Infinite dimensional Hilbert spaces over $\mathbb{H}$ were called *Wachs spaces* (German: *Wachssche Räume*), after his fellow student with that name who inspired him to this research.

Now we are intrested over similar things over the octonions $\mathbb{O}$, but only *finite* dimensions.

Definition 4 *By a left Wachs module over the octonions $\mathbb{O}$ we mean a free finitely generated $\mathbb{O}$-module equipped with an inner product $M \times M \to \mathbb{O} : (x, y) \mapsto \langle x, y \rangle$ such that there exists an orthonormal basis $e_1, \ldots . e_n$ such that*

$$\langle x, y \rangle = \sum_k x^k \overline{y^k}.$$

Remark 5 *In the case of* right *Wachs modules the inner product is instead defined by*

$$\langle x, y \rangle = \sum_k \overline{y^k} x^k.$$

Actually, in [3], Teichmüller works with right *modules, not, as we, with left ditto.*

I have been to lazy to learn all details of Teichmüller's proof in [3]. However, it appears that an essential ingredient there is the presence of a *bimodule* structure on the given right module M, that is, besides the given right multipliaction $(\lambda, x) \mapsto x\lambda$ one, a left $(\lambda, x) \mapsto \lambda x$. For a fixed $\lambda \in \mathbb{H}$, let $L(\lambda)$ be the operator of left multiplication, $L(\lambda x) = \lambda x$. Teichmüller also imposes the supplementary condition

$$(L(\lambda))^\star = L(\overline{\lambda}), \tag{14}$$

where the star stands for the operator adjoint and the bar for conjugation of quaternions. In this sitation the author speaks of a *quaternion representation* (German: "eine Quaternionendarstellung"). Given a quaternion representation L the author shows that there exists in M an orthogonal basis $e_1, \ldots .e_n$ such that

$$L(\lambda)e_k = e_k\lambda.$$

Returning to the case of octonions, the analogue of (14) still make sense, but is not yet clear to me if Teichmüller's above theorem extends.

One can also wonder what happens if one passes further on in the *Cayley-Dickson hierarchy.* After the octonions $\mathbb{O}$ comes the 16-dimensional algebra of *sedenions* [1], which has not even got an established notation in mathematics. So, does the present procedure extend to sedenion modules as well?

Referenser

[1] Jaak Lõhmus - Eugene Paal - Leo Sorgsepp: Nonassociative algebras in physics. Hadronic Press Monographs in Mathematics. Hadronic Press, Inc., Palm Harbor, FL, 1994.

[2] Jaak Peetre: On quaternion bimodules and quaternion trilinear forms. TEX-script.

[3] Oswald Teichmüller: Operatoren im Wachsschen Raum. J. Reine Angew. Math. 274 (1935), 74-124.

[4] Bartel Leendert van der Waerden: Algebra II. Vierte Auflage. (Die Grundlehren der mathematischen Wissenschaften 34.) Springer-Verlag, Berlin - Göttingen - Heidelberg, 1959.