An integration problem For Jonathan, Fernando, Miroslav, Thomas, Hjalmar, Genkai and other friends, Reprint of 1996 version file: intprob.tex

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Mathematical discoveries glimpsed and lost to view will have their time again. Tom Stoppard, Arcadia.

1 Background

I have been interested in multi- and in particular trilinear forms for nearly three decades. I was thereby inspired by the bold program of Albrecht Pietsch at a meeting in Leipzig [1][1]. In particular, I was led to the study the analogue of *Shatten-Neumann classes* $\mathcal{S}_p$ for opertors in Hilbert spaces in the multilinear situation.[2] But the difficulties turned out to be insurmountable, so I decided to work first with the seemingly simpler problem of describing bounded trilinear forms of low dimensions. In the case or binary trilinear forms over $\mathbb{C}$, which problem was settled completely [2]. I was guided by the idea that the set of binary forms of norm unity – *the trilinear ball* –

[1]I did not participate in it myself, but I went to see him and Hans Triebel in Jena little later. However, one day these gentlemen, not having time for me, I was sent to Weimar, and the sight of the illfamed concentration camp Buchenwald. Weimar is the the pinnacle of German civilization and a few miles from it this place of horror. The camp was first used by the Nazis and, after the war, by the Sovjets.

[2]For a first, rather primitive mention of these see [3].

is the analogue of a *bounded symmetric domain* – because in the analogous situation with bilinear forms one gets precisely such domains.[3] Trying to develop harmonic analysis in the the trilinear ball, one has in the first place to determine its volume, to which problem this paper is devoted. It was written a long time ago, when I was not as stupid as I am today. It is still mainly unsolved, but perhaps I could inspire some of you to have a look at it. Even a numerical result might be of interest.

2 The integral

Let x_1, x_2, x_3 be positive variables and put

$$S = x_1^2 + x_2^2 + x_3^2, \quad P = x_1 x_2 x_3.$$

Let $\mathfrak{E}$ be the open set $\{S + P < 1\}$ and consider the integral

$$I_{ab} = \int_{\mathfrak{E}} S^a P^{2b} x_1 x_2 x_3 \, dx_1 dx_2 dx_3,$$

where a and b are integers > 0.

We have computed I_{ab} for low values of a and b using Mathematica. The results are depicted in the chart on next page.

From this experience we can make a number of observations. It seems that I_{ab} is always a fraction of the form $\frac{A-B\pi^2}{C}$, where A, B, C are positive integers. In addition, it seems that the fraction $\frac{A}{B}$ is a close approximation π^2. For instance, if $a = 1$, $b = 3$ the difference between these two numbers is about

$$1,57067 \cdot 10^{-10}.$$

Another interesting fact is that if one has $3I_{02} = I_{40}$ or $3\int P^2 = \int S^4$. Are there other such relations? If some, find all of them.

I have likewise observerd that if one takes $I = I_{00}$ and $J = I_{00}$ as a basis, forgetting the denomiunators, then the remaining integrals I_{ab} seem to come as *halfinteger* combinations of the former.

Example:

$$I_{13} = -\frac{67230837}{2} I + \frac{1777043}{2} J.$$

Kåseberga, March 14, 1996.

[3]As an example of a bounded symmetric domain we have the unit disk in i $\mathbb{C}$ or its "dual", the Riemann sphere.

List of I_{ab} for $a = 0, 1, 2, 3, 4$, $b = 2, 4, 6$ and S_5.

$$
\begin{aligned}
\int &= \frac{16 - \pi^2}{512} \\
\int P^2 &= \frac{512 - 51\pi^2}{49152} \\
\int P^4 &= \frac{131072 - 13275\pi^2}{11050200} \\
\int P^6 &= \frac{607649792 - 61567275\pi^2}{30281134499} \\
\int S &= \frac{112 - \pi^2}{3072} \\
\int SP^2 &= \frac{8192 - 825\pi^2}{409600} \\
\int SP^4 &= \frac{117888821248 - 1802239005053\pi^2}{1790498045952} \\
\int SP^6 &= \frac{243857904390144 - 2470500023416425\pi^2}{40345889302118400} \\
\int S^2 &= \frac{1280 - 117\pi^2}{24576} \\
\int S^2P^2 &= \frac{398336 - 40275\pi^2}{9830400} \\
\int S^2P^4 &= \frac{45827712 - 46426275\pi^2}{97707033600} \\
\int S^2P^6 &= \frac{12531609216 - 12607724025\pi^2}{11009812838400} \\
\int S^3 &= \frac{210944 - 20475\pi^2}{2457600} \\
\int S^3P^2 &= \frac{309246 - 31311\pi^2}{3612672} \\
\int S^3P^4 &= \frac{229521227776 - 2325506575\pi^2}{1664719257600} \\
\int S^3P^6 &= \frac{46361485508608 - 4697399512575\pi^2}{167859191908000} \\
\int S^4 &= \frac{5(512 - 51)\pi^2}{16384}4 \\
\int S^4P^2 &= \frac{961150976 - 97361775\pi^2}{5138022400} \\
\int S^4P^4 &= \frac{180808056832 - 18319614075\pi^2}{554906419200} \\
\int S^4P^6 &= \frac{293874314009056 - 20775691331475\pi^2}{429795310284800} \\
\int S^5 &= \frac{49030144 - 4935525\pi^2}{160563200}
\end{aligned}
$$

3 Summary

We investigate the integral $I_{ab} = \int_{\mathfrak{E}} S^a P^{2b} x_1 x_2 x_3 \, dx_1 dx_2 dx_3$, extended over the set $\mathfrak{E} = \{S = x_1^2 + x_2^2 + x_3^2, P = x_1 x_2 x_3\}$, where a and b are integers > 0. The problem (yet unsolved) arose long ago (1996) in connection with binary trilinear forms over $\mathbb{C}$ (confere [2]).

References

[1] Albrecht Pietsch: Ideals of Multilinear Functionals (Designs of aTheory). Proceedings of the second international conference on operator algebras, ideals, and their applications in theoretical physics (Leipzig, 1983), 185–199, Teubner-Texte Math., 67, Teubner, Leipzig, 19

[2] Fernando Cobos – Thomas Kühn– Jaak Peetre: Extreme Points of the Complex Trilinear Ball. Studia Math. 138 (2000), 81-92.

[3] Jaak Peetre: Paracommutators and Minimal Spaces. In: S.C. Power (ed.), Operators and FunctionTheory, pp. 163-224. Reidel, Dordrecht, 1985.