ON THE SOLVABILITY OF SYSTEMS OF PSEUDODIFFERENTIAL OPERATORS

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Dedicated to Hans Duistermaat on his sixtyfifth birthday

1. Introduction

In this paper we shall study the question of local solvability for square systems of classical pseudodifferential operators $P \in \Psi^m_{cl}(M)$ on a C^∞ manifold M. We shall only consider operators acting on distributions $\mathcal{D}'(M, \mathbf{C}^N)$ with values in $\mathbf{C}^N$ but since the results are local and invariant under base changes, they immediately carry over to operators on sections of vector bundles. We shall assume that the symbol of P is an asymptotic sum of homogeneous $N \times N$ matrices, with homogeneous principal symbol $p = \sigma(P)$. We shall also assume that P is of principal type, so that the principal symbol vanishes of first order on the kernel, see Definition 2.1.

Local solvability for a $N \times N$ system P at a compact set $K \subseteq M$ means that the equation

$$Pu = v \tag{1.1}$$

has a local weak solution $u \in \mathcal{D}'(M, \mathbf{C}^N)$ in a neighborhood of K for all $v \in C^\infty(M, \mathbf{C}^N)$ in a subset of finite codimension. We say that P is microlocally solvable at a compactly based cone $K \subset T^*M$ if there exists an integer N such that for every $f \in H^{loc}_{(N)}(M, \mathbf{C}^N)$ there exists $u \in \mathcal{D}'(M, \mathbf{C}^N)$ so that $K \bigcap \mathrm{WF}(Pu - f) = \emptyset$, see [14, Definition 26.4.3]. Here $H_{(s)}$ is the usual L^2 Sobolev space and $H^{loc}_{(s)}$ is the localized Sobolev space, i.e., those $f \in \mathcal{D}'$ such that $\phi f \in H_{(s)}$ for any $\phi \in C_0^\infty$.

Hans Lewy's famous counterexample [25] from 1957 showed that not all smooth linear differential operators are solvable. It was conjectured by Nirenberg and Treves [28] in 1970 that local solvability for principal type *scalar* pseudodifferential operators is equivalent to

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Date: February 25, 2010.

2000 *Mathematics Subject Classification.* 35S05 (primary) 35A07, 47G30, 58J40 (secondary).

condition (Ψ) on the principal symbol p, which means that

(1.2) $\operatorname{Im}(ap)$ does not change sign from $-$ to $+$

along the oriented bicharacteristics of $\operatorname{Re}(ap)$

for any $0 \neq a \in C^\infty(T^*M)$. Recall that the operator is of principal type if $dp \neq 0$ when $p = 0$, and the oriented bicharacteristics are the positive flow-outs of the Hamilton vector field

$$H_{\operatorname{Re}(ap)} = \sum_j \partial_{\xi_j} \operatorname{Re}(ap)\partial_{x_j} - \partial_{x_j} \operatorname{Re}(ap)\partial_{\xi_j}$$

on $\operatorname{Re}(ap) = 0$ (also called the semibicharacteristics of p). Condition (1.2) is obviously invariant under symplectic changes of coordinates and multiplication with non-vanishing factors. Thus the condition is invariant under conjugation of P with elliptic Fourier integral operators. It actually suffices to check the condition with some $0 \neq a \in C^\infty$ such that $d(\operatorname{Re} ap) \neq 0$, see [14, Lemma 26.4.10]. Recall that p satisfies condition $(\overline{\Psi})$ if $\overline{p}$ satisfies condition (Ψ), and that p satisfies condition (P) if there are no sign changes on the semibicharacteristics, that is, p satisfies both condition (Ψ) and $(\overline{\Psi})$.

The necessity of (Ψ) for local solvability of scalar pseudodifferential operators was proved by Moyer [27] in 1978 for the two dimensional case, and by Hörmander [13] in 1981 for the general case. The sufficiency of condition (Ψ) for solvability of scalar pseudodifferential operators in two dimensions was proved by Lerner [18] in 1988. The Nirenberg-Treves conjecture was finally proved by the author [8], giving solvability with a loss of two derivatives (compared with the elliptic case). This has been improved to a loss of arbitrarily more than 3/2 derivatives by the author [9], and to a loss of exactly 3/2 by Lerner [24]. Observe that there only exist counterexamples showing a loss of $1 + \varepsilon$ derivatives for arbitrarily small $\varepsilon > 0$, see Lerner [19].

For partial differential operators, condition (Ψ) is equivalent to condition (P). The sufficiency of (P) for local solvability of scalar partial differential operators with a loss of one derivative was proved in 1973 by Beals and Fefferman [1], introducing the Beals-Fefferman calculus. In the case of operators which are *not* of principal type, conditions corresponding to (Ψ) are neither necessary nor sufficient for local solvability, see [3].

For systems there is no corresponding conjecture for solvability. By looking at diagonal systems, one finds that condition (Ψ) for the eigenvalues of the principal symbol is necessary for solvability. But when the principal symbol is not diagonalizable, condition (Ψ) is not sufficient, see Example 2.14 below. It is not even known if condition (Ψ) is sufficient in the case when the principal symbol is C^∞ diagonalizable. We shall consider the case of when the principal symbol has constant characteristics, then the eigenvalue close to the origin has constant multiplicity, see Definition 2.5. In that case, the eigenvalue is a C^∞ function and condition (Ψ) on the eigenvalues is well-defined. The main result of the paper is that classical square systems of pseudodifferential operators of principal type having constant characteristics are solvable (with a loss of 3/2 derivatives) if and only if the eigenvalues of the principal symbol satisfies condition (Ψ), see Theorem 2.7.

2. Statement of results

We say that the system $P \in \Psi^m_{cl}$ is *classical* if the symbol of P is an asymptotic sum $P_m + P_{m-1} + \dots$ where $P_j(x,\xi)$ is homogeneous of degree j in ξ, here P_m is called the *principal symbol* of P. Recall that the eigenvalues of the principal symbol are the solutions to the characteristic equation

$$|P_m(x,\xi) - \lambda \operatorname{Id}_N| = 0$$

where $|A|$ is the determinant of the matrix. In the following, we shall denote by $\operatorname{Ker} A$ the kernel and $\operatorname{Ran} A$ the range of the matrix A. The definition of principal type for systems is similar to the one for scalar operators.

Definition 2.1. We say that the $N \times N$ system $P(w) \in C^1$ is of *principal type* at w_0 if

$$\partial_\nu P(w_0) : \operatorname{Ker} P(w_0) \mapsto \operatorname{Coker} P(w_0) = \mathbf{C}^N / \operatorname{Ran} P(w_0) \tag{2.1}$$

is bijective for some ν, here $\partial_\nu P = \langle \nu, dP \rangle$ and the mapping (2.1) is given by $u \mapsto \partial_\nu P(w_0)u$ modulo $\operatorname{Ran} P(w_0)$. We say that $P \in \Psi^m_{cl}$ is of principal type at w_0 if the principal symbol $P_m(x,\xi)$ is of principal type at w_0.

Remark 2.2. *If $P(w) \in C^1$ is of principal type and $A(w)$, $B(w) \in C^1$ are invertible then APB is of principal type. We also have that P is of principal type if and only if the adjoint P^* is of principal type.*

In fact, Leibniz' rule gives

$$d(APB) = (dA)PB + A(dP)B + APdB \tag{2.2}$$

and $\operatorname{Ran}(APB) = A(\operatorname{Ran} P)$ and $\operatorname{Ker}(APB) = B^{-1}(\operatorname{Ker} P)$ when A and B are invertible, which gives the invariance under left and right multiplication. Since $\operatorname{Ker} P^*(w_0) =$

$\operatorname{Ran} P(w_0)^\perp$ we find that P satisfies (2.1) if and only if

$$\operatorname{Ker} P(w_0) \times \operatorname{Ker} P^*(w_0) \ni (u,v) \mapsto \langle \partial_\nu P(w_0)u, v\rangle \tag{2.3}$$

is a non-degenerate bilinear form. Since $\langle \partial_\nu P^* u, v\rangle = \overline{\langle \partial_\nu P v, u\rangle}$ we then obtain that P^* is of principal type.

Observe that only square systems can be of principal type since

$$\operatorname{Dim}\operatorname{Ker} P = \operatorname{Dim}\operatorname{Coker} P + M - N$$

if P is an $N \times M$ system. In general, if the system is of principal type and has constant multiplicity of the eigenvalues then there are no non-trivial Jordan boxes, see Definition 2.3 and Proposition 2.10. Then we also have that the eigenvalues λ are of principal type: $d\lambda \neq 0$ when $\lambda = 0$. When the multiplicity is equal to one, this condition is sufficient. In fact, by using the spectral projection one can find invertible systems A and B so that

$$APB = \begin{pmatrix} \lambda & 0 \\ 0 & E \end{pmatrix}$$

with E invertible $(N-1)\times(N-1)$ system, and this system is obviously of principal type.

Definition 2.3. Let A be an $N\times N$ matrix and λ an eigenvalue of A. The multiplicity of λ as a root of the characteristic equation $|A - \lambda \operatorname{Id}_N| = 0$ is called the *algebraic* multiplicity of the eigenvalue, and the dimension of $\operatorname{Ker}(A - \lambda \operatorname{Id}_N)$ is called the *geometric* multiplicity.

Observe that if the matrix $P(w)$ depend continuously on a parameter w, then the eigenvalues $\lambda(w)$ also depend continuously on w. Such a continuous function $\lambda(w)$ of eigenvalues we will call a *section of eigenvalues of* $P(w)$.

Remark 2.4. *If the section of eigenvalues $\lambda(w)$ of the $N \times N$ system $P(w) \in C^\infty$ has constant* algebraic *multiplicity then $\lambda(w) \in C^\infty$. In fact, if k is the multiplicity then $\lambda = \lambda(w)$ solves $\partial_\lambda^{k-1}|P(w) - \lambda \operatorname{Id}_N| = 0$ so $\lambda(w) \in C^\infty$ by the Implicit Function Theorem.*

This is *not* true for constant geometric multiplicity, for example $P(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$, $t \in \mathbf{R}$, has geometric multiplicity equal to one for the eigenvalues $\pm\sqrt{t}$. Observe the geometric multiplicity is lower or equal to the algebraic, and for symmetric systems they are equal. We shall assume that the eigenvalues close to zero have constant algebraic and geometric multiplicities by the following definition.

Definition 2.5. The $N \times N$ system $P(w) \in C^\infty$ has *constant characteristics* near w_0 if there exists an $\varepsilon > 0$ so that any section of eigenvalues $\lambda(w)$ of $P(w)$ with $|\lambda(w)| < \varepsilon$ has both constant algebraic and geometric multiplicity when $|w - w_0| < \varepsilon$.

Definition 2.5 is invariant under changes of bases: $P \mapsto E^{-1}PE$ where E is an invertible system, since this preserves the multiplicities of the eigenvalues of the system. It is also invariant under taking adjoints, since $|P^*(w) - \lambda^*(w)\operatorname{Id}| = \overline{|P(w) - \lambda(w)\operatorname{Id}|}$ and $\operatorname{Dim}\operatorname{Ker}(P^*(w) - \lambda^*(w)\operatorname{Id}) = \operatorname{Dim}\operatorname{Ker}(P(w) - \lambda(w)\operatorname{Id})$. The definition is *not* invariant under multiplication of the system with invertible systems, even in the case when $P(w) = \lambda(w)\operatorname{Id}$ since $A(w)P(w) = \lambda(w)A(w)$ need not have constant characteristics.

Observe that generically the eigenvalues of a system have constant multiplicity, but not necessarily when equal to zero. For example, the system

$$P(w) = \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix}$$

is symmetric and of principal type with eigenvalues $\pm\sqrt{w_1^2 + w_2^2}$, which have constant multiplicity except when equal to 0.

Definition 2.6. Let the $N \times N$ system $P \in \Psi^m_{cl}$ be of principal type and constant characteristics. We say that P satisfies condition (Ψ) or (P) if the eigenvalues of the principal symbol satisfies condition (Ψ) or (P).

Observe that the eigenvalue close to the origin is a uniquely defined C^∞ function of principal type by Definition 2.3 and Proposition 2.10. Thus, the semibicharacteristics of the eigenvalues are well-defined near the characteristic set $\{\, w : \ |P(w)| = 0 \,\}$, so the conditions (Ψ) and (P) on the eigenvalues are well-defined. Also well-defined is the condition that the Hamilton vector field of an eigenvalue λ does not have the radial direction when $\lambda = 0$.

To get local solvability at a point $x_0 \in M$ we shall also assume a strong form of the non-trapping condition at x_0 for the eigenvalues λ of P:

$$\lambda = 0 \implies \partial_\xi \lambda \neq 0 \tag{2.4}$$

This means that all non-trivial semibicharacteristics of λ are transversal to the fiber $T^*_{x_0}M$, which originally was the condition for principal type of Nirenberg and Treves [28]. Microlocally, in a conical neighborhood of a $(x, \xi) \in T^*M$, we can always obtain (2.4) after a canonical transformation. In the following, we shall use the usual L^2 Sobolev norm $\|u\|_{(s)}$ and the L^2 norm $\|u\| = \|u\|_{(0)}$.

Theorem 2.7. *Let $P \in \Psi^m_{cl}(M)$ be an $N \times N$ system of principal type and constant characteristics near $(x_0, \xi_0) \in T^*M$, such that the Hamilton vector field of an eigenvalue λ does not have the radial direction when $\lambda = 0$. Then P is microlocally solvable near (x_0, ξ_0) if and only if condition (Ψ) is satisfied near (x_0, ξ_0), and then*

$$\|u\| \le C(\|P^*u\|_{(3/2-m)} + \|Ru\| + \|u\|_{(-1)}) \qquad u \in C_0^\infty(M, \mathbf{C}^N) \tag{2.5}$$

Here $R \in \Psi^{1/2}_{1,0}(M)$ is a $K \times N$ system such that $(x_0, \xi_0) \notin \mathrm{WF}(R)$, which gives microlocal solvability of P at (x_0, ξ_0) with a loss of at most $3/2$ derivatives. If the eigenvalues also satisfy (2.4) at $x_0 \in M$, then we obtain (2.5) with $x \neq x_0$ in $\mathrm{WF}(R)$, which gives local solvability of P at x_0 with a loss of at most $3/2$ derivatives.

As usual, $\mathrm{WF}(R)$ is the smallest smallest conical set in $T^*M \setminus 0$ such that $R \in \Psi^{-\infty}$ in the complement. The conditions in Theorem 2.7 are invariant under conjugation with scalar Fourier integral operators since they only depend on the principal symbol of the system. They are also invariant under the base change: $P \mapsto E^{-1}PE$ with invertible system E, since this preserves the eigenvalues of the principal symbol. The conditions of Theorem 2.7 are more or less necessary, of course condition (Ψ) is necessary even in the scalar case. Example 2.14 shows that the condition of principal type is necessary in the case of constant characteristics, and Example 2.15 shows that constant characteristics is necessary for solvability for systems of principal type.

We shall postpone the proof of Theorem 2.7 to Section 4. The proof of the necessity is essentially the classical Moyer-Hörmander proof for the scalar case. The proof of the sufficiency will be an adaption of the proof for the scalar case in [8], using some of the ideas of Lerner [24]. In fact, since the normal form of the operator will have a scalar principal symbol, the multiplier will essentially be the same as in [8]. But since we lose more than one derivative in the estimate we also have to consider the lower order matrix valued terms in the expansion of the operator. This is done in Section 7 and is the main new part of the paper. In Section 3 we review the Weyl calculus and state the estimates we will use in the proof of Theorem 2.7. But we shall postpone the proof of the semiclassical estimate of Proposition 3.6 until Section 7. In Section 4 we prove Theorem 2.7 by a microlocal reduction to a normal form using the estimates in Section 3. In Section 5 we define the symbol classes and weights we are going to use. In Section 6 we review the Wick quantization, introduce the function spaces and the multiplier estimate that we will use for the proof of Proposition 3.6. Finally, in Section 7 we prove Proposition 3.6

by estimating the contributions of the lower order terms. The proof of Theorem 2.7 in Section 4 also gives the following results.

Remark 2.8. *If P is of principal type with constant characteristics satisfying condition (P) then we get the estimate* (2.5) *with* $3/2$ *replaced by* 1*. If P satisfies condition* $(\overline{\Psi})$ *and some repeated Poisson bracket of the real and imaginary parts of the eigenvalue close to the origin is non-vanishing, then we obtain a subelliptic estimate for P with* $3/2$ *replaced by* $k/k+1$ *in* (2.5) *for some* $k \in \mathbf{Z}_+$*, see* [14, Chapter 27].

The Poisson bracket of f and g is defined by $\{ f, g \} = H_f g$. Theorem 2.7 has applications to scalar non-principal type pseudodifferential operators by the following result.

Theorem 2.9. *Let $Q \in \Psi^1_{cl}(M)$ be a scalar operator of principal type near $(x_0, \xi_0) \in T^*M$ and let $A_j \in \Psi^0_{cl}(M)$, $j = 1, \dots, N$ be scalar. Then the equation*

$$Pu = Q^N u + \sum_{j=0}^{N-1} A_j Q^j u = f \tag{2.6}$$

is locally solvable near (x_0, ξ_0) if and only if $\sigma(Q)$ satisfies condition (Ψ) near (x_0, ξ_0).

Proof. This is a standard reduction to a first order system. For scalar $u \in \mathcal{D}'$ we let $u_{j+1} = Q^j u$ for $0 \le j < N$. Then (2.6) holds if and only if $U = {}^t(u_1, \dots, u_N)$ solves

$$\mathbb{P} U = F \tag{2.7}$$

where

$$\mathbb{P} = \begin{pmatrix} Q & -1 & 0 & 0 & \dots \\ 0 & Q & -1 & 0 & \dots \\ 0 & 0 & Q & -1 & \dots \\ \dots \\ A_0 & A_1 & A_2 & \dots & Q + A_{N-1} \end{pmatrix}$$

and $F = {}^t(0, 0, \dots, f)$. Now the equation (2.6) is locally solvable if and only if the system (2.7) is locally solvable. In fact, to solve (2.7) we first put $u_1 = 0$, $u_2 = -f_1$ and recursively $u_{j+1} = Qu_j - f_j$ for $1 \le j < N$. Then we only have to solve (2.6) for $u = v_1$ with f depending on f_j, and add $Q^{j-1} v_1$ to u_j. Now $\sigma(\mathbb{P}) = \sigma(Q)\,\mathrm{Id}_N$ which is of principal type with constant characteristics so it is locally solvable if and only if Q satisfies condition (Ψ) according to Theorem 2.7. □

We shall conclude the section with some examples. But first we prove a result about the characterization of systems of principal type.

Proposition 2.10. *Assume that $P(w) \in C^\infty$ is an $N \times N$ system such that $|P(w_0)| = 0$ and there exists an $\varepsilon > 0$ such that the eigenvalue λ of $P(w)$ with $|\lambda| < \varepsilon$ has constant algebraic multiplicity in a neighborhood of w_0. Let $\lambda(w) \in C^\infty$ be the unique eigenvalue for $P(w)$ near w_0 satisfying $\lambda(w_0) = 0$ by Remark 2.4. Then $P(w)$ is of principal type at w_0 if and only if $d\lambda(w_0) \neq 0$ and the geometric multiplicity of the eigenvalue λ is equal to the algebraic multiplicity at w_0.*

Thus, if $P(w)$ is of principal type having constant characteristics, then all sections of eigenvalues $\lambda(w)$ are of principal type and we have no non-trivial Jordan boxes in the normal form. This means that for symmetric systems having constant characteristics it suffices that the eigenvalues are of principal type. If $P(w)$ does not have constant characteristics then this is no longer true, in fact the eigenvalues need not even be differentiable, see Example 2.16.

Observe that if $P(w)$ is of principal type and has constant characteristics, then $P(w) - \lambda \operatorname{Id}_N$ is of principal type near w_0 for $|\lambda| \ll 1$. In fact, the algebraic and geometric multiplicities are constant for the eigenvalue λ and $d\lambda \neq 0$ near w_0.

Now the eigenvalue $\lambda(w)$ in Proposition 2.10 is the unique C^∞ solution to $\partial_\lambda^{k-1}|P(w) - \lambda \operatorname{Id}_N| = 0$ according to Remark 2.4, where k is the algebraic multiplicity. Thus we find that $d\lambda(w) \neq 0$ if and only if

$$\partial_w \partial_\lambda^{k-1}|P(w) - \lambda \operatorname{Id}_N| \neq 0 \qquad \text{when } \lambda = \lambda(w)$$

We only need this condition for a symmetric systems having constant multiplicity to be of principal type.

Example 2.11. Let

$$P(w) = \begin{pmatrix} w_1 + iw_2^2 & w_2 \\ 0 & w_1 + iw_2^2 \end{pmatrix} \qquad w = (w_1, w_2) \in \mathbf{R}^2$$

then P is of principal type, has constant algebraic multiplicity of the eigenvalue $w_1 + iw_2^2$ but not constant geometric multiplicity. In fact, $\partial_{w_1} P = \operatorname{Id}_2$, $P(w)$ has non-trivial kernel only when $w_2 = 0$ but the geometric multiplicity of the eigenvalue is equal to one when $w_2 \neq 0$.

Example 2.12. Let

$$P = p(x, D_x) \operatorname{Id}_N + B(x, D_x) + P_0(x, D_x)$$

where $p \in S^1_{cl}$ is a scalar homogeneous symbol of principal type, $B \in \Psi^1_{cl}$ with nilpotent homogeneous principal symbol $\sigma(B)$ and $P_0 \in \Psi^0_{cl}$. Then p is the only eigenvalue to $\sigma(P)$ and P is of principal type if and only if $\sigma(B) = 0$ when $p = 0$ by Proposition 2.10.

Remark 2.13. *Observe that the conclusion of Proposition 2.10 does not hold if the algebraic multiplicity is not constant. For example*

$$P(w) = \begin{pmatrix} w_1 & 1 \\ w_2 & w_1 \end{pmatrix} \qquad w = (w_1, w_2) \in \mathbf{R}^2$$

has determinant is equal to $w_1^2 - w_2$ *and eigenvalues* $w_1 \pm \sqrt{w_2}$*, so the geometric but not the algebraic multiplicity is constant near* $w_2 = 0$*. Since*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P(w) = \begin{pmatrix} w_2 & w_1 \\ w_1 & 1 \end{pmatrix}$$

we find that $P(w)$ *is of principal type at* $(0,0)$ *by the invariance.*

Proof of Proposition 2.10. First we note that $P(w)$ is of principal typ at w_0 if and only if

$$\partial_\nu^k |P(w_0)| \neq 0 \qquad k = \operatorname{Dim}\operatorname{Ker} P(w_0) \tag{2.8}$$

for some $\nu \in T(T^*\mathbf{R}^n)$. Observe that $\partial^j |P(w_0)| = 0$ for $j < \operatorname{Dim}\operatorname{Ker} P(w_0)$. In fact, by choosing bases for $\operatorname{Ker} P(w_0)$ and $\operatorname{Im} P(w_0)$ respectively, and extending to bases of $\mathbf{C}^N$, we obtain matrices A and B so that

$$AP(w)B = \begin{pmatrix} P_{11}(w) & P_{12}(w) \\ P_{21}(w) & P_{22}(w) \end{pmatrix}$$

where $|P_{22}(w_0)| \neq 0$ and P_{11}, P_{12} and P_{21} all vanish at w_0. By the invariance, P is of principal type if and only if $\partial_\nu P_{11}$ is invertible for some ν, so by expanding the determinant we obtain (2.8).

Now since the eigenvalue $\lambda(w)$ has constant algebraic multiplicity near w_0, we find that

$$|P(w) - \lambda \operatorname{Id}_N| = (\lambda(w) - \lambda)^m e(w, \lambda)$$

near w_0, where $\lambda(w_0) = 0$, $e(w, \lambda) \neq 0$ and $m \geq \operatorname{Dim}\operatorname{Ker} P(w_0)$ is the algebraic multiplicity. By putting $\lambda = 0$ we obtain that $\partial_\nu^j |P(w_0)| = 0$ if $j < m$ and $\partial_\nu^m |P(w_0)| = (\partial_\nu \lambda(w_0))^m e(w_0, 0)$ which proves Proposition 2.10. ☐

The following example shows that if the system is not of principal type then it need not be solvable, even if it has real eigenvalues with constant characteristics.

Example 2.14. Let $P \in \Psi^2_{cl}$ have principal symbol $\sigma(P) = p^2$ where $p \in S^1_{cl}$ is real, homogeneous of degree 1, and of principal type. Then P is not solvable if the imaginary part of the subprincipal symbol of P changes sign on the bicharacteristics of p by [29]. Observe that the subprincipal symbol is invariantly defined at the double characteristics $p^{-1}(0)$. As in the proof of Theorem 2.9, the equation can be reduced to the system

$$\mathbb{P} = \begin{pmatrix} p(x, D_x) & p_1(x, D_x) \\ -1 & p(x, D_x) \end{pmatrix}$$

where $\sigma(p_1) \in S^1_{cl}$ is equal to the subprincipal symbol of P on $p^{-1}(0)$. The system is of principal type at $w \in p^{-1}(0)$ if and only if $\sigma(p_1)(w) = 0$ by Proposition 2.10. This system has real eigenvalues of constant characteristics so it satisfies condition (P), but it is neither solvable nor of principal type if $\operatorname{Im} p_1$ changes sign along the bicharacteristic of p. Observe that the system

$$\mathbb{P} = \begin{pmatrix} p(x, D_x) & p_1(x, D_x) \\ 0 & p(x, D_x) \end{pmatrix}$$

is solvable, since it is upper triangle with solvable diagonal elements. Thus the solvability depends on the lower order terms in this case.

The next example is an unsolvable system of principal type with real eigenvalues, but it does not have constant characteristics.

Example 2.15. Let

$$P = \begin{pmatrix} D_{x_1} & B(x, D_x) \\ -1 & D_{x_1} + R(D_x) \end{pmatrix} \tag{2.9}$$

where $R(\xi) = \xi_2^2/|\xi|$ and $\sigma(B)(x, \xi) = \xi_2 B_0(x, \xi)$ with homogeneous $B_0 \in S^0$. The eigenvalues of the principal symbol $\sigma(P)$ are ξ_1 and $\xi_1 + R(\xi)$ which are real and coincide when $\xi_2 = 0$. Since $\partial_{\xi_1}\sigma(P) = \mathrm{Id}_2$ and $\sigma(B)$ vanish when $\xi_2 = 0$, we find that P is of principal type by Proposition 2.10. If $t \mapsto \operatorname{Im} B_0(t, x', 0, 0, \xi'')$ changes sign at $t = x_1$, then P is not microlocally solvable at $(x, 0, 0, \xi'')$, here $x = (x_1, x') = (x_1, x_2, x'')$ and $\xi'' \neq 0$. In fact, the system $PU = F$ with $U = {}^t(u_1, u_2)$ and $F = {}^t(f_1, f_2)$ is equivalent to the equation

$$Qu_2 = (D_{x_1}(D_{x_1} + R(D_x)) + B(x, D_x))u_2 = f_1 + D_{x_1} f_2 \tag{2.10}$$

if we put $u_1 = (D_{x_1} + R(D_x))u_2 - f_2$. Thus the system P is solvable if and only if Q is solvable. That Q is not solvable follows from using the construction of approximate solutions to the adjoint in [26], replacing D_{x_2} with $R(D_x)$.

We can also generalize this to the case where

$$R(\xi) = \xi_2^k |\xi|^{1-k}$$

and $\sigma(B)(x,\xi) = \xi_2^j B_j(x,\xi)$ with $j < k$ and $B_j \in S^{1-j}$ homogeneous, satisfying the same conditions as B_0. On the other hand, if $\sigma(B)(x,\xi) = \xi_2^j B_j(x,\xi)$ with $j \ge k$ then we can write

$$B(x, D_x) \cong A(x, D_x)R(D_x) \qquad \text{modulo } \Psi^0$$

for some $A \in \Psi^0$ and then

$$\begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix} P \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \cong \begin{pmatrix} D_{x_1} & 0 \\ 0 & D_{x_1} + R(D_x) \end{pmatrix} \qquad \text{modulo } \Psi^0$$

which is solvable. In fact, the principal symbol is on diagonal form with real diagonal elements of principal type, giving L^2 estimates of the adjoint which can be perturbed by lower order terms.

Finally, we have an example of an unsolvable operator which is diagonalizable and self-adjoint, but not of principal type.

Example 2.16. Take real $b(t) \in C^\infty(\mathbf{R})$, and define the symmetric system

$$P = \begin{pmatrix} D_t + b(t)D_x & (t - ib(t))D_x \\ (t + ib(t))D_x & -D_t + b(t)D_x \end{pmatrix} = P^* \qquad (t, x) \in \mathbf{R}^2$$

Eigenvalues of $\sigma(P)$ are $b(t)\xi \pm \sqrt{\tau^2 + (t^2 + b^2(t))\xi^2}$ which are zero for $(\tau, \xi) \ne 0$ only if $t = \tau = 0$. The eigenvalues coincide for $(\tau, \xi) \ne 0$ if and only if $b(t) = t = \tau = 0$. We have that

$$Q = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} P \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} D_t - itD_x & 2b(t)D_x \\ 0 & D_t + itD_x \end{pmatrix}$$

which is not locally solvable at $t = 0$ for any choice of $b(t)$, since $D_t + itD_x$ is not locally solvable, condition (Ψ) is not satisfied when $\xi > 0$. The eigenvalues of the principal symbol $\sigma(Q)$ are $\tau \pm it\xi$. By the invariance, P is of principal type if and only if $b(0) = 0$. When $b(t) \ne 0$ we find that $\sigma(P)$ is diagonalizable and self-adjoint, but not of principal type. When $b \equiv 0$ the system is symmetric of principal type, but does not have constant characteristics.

3. The multiplier estimates

In this section we shall prove multiplier estimates for microlocal normal forms of the adjoint operator, which we shall use in the proof of Theorem 2.7. We shall consider the model operators

$$P_0 = (D_t + iF(t, x, D_x)) \operatorname{Id}_N + F_0(t, x, D_x) \tag{3.1}$$

where $F \in C^\infty(\mathbf{R}, \Psi^1_{cl}(\mathbf{R}^n))$ is scalar with with real homogeneous principal symbol $\sigma(F) = f$, and $F_0 \in C^\infty(\mathbf{R}, \Psi^0_{cl})$ is an $N \times N$ system. In the following, we shall assume that P_0 satisfies condition $(\overline{\Psi})$:

$$f(t, x, \xi) > 0 \quad \text{and } s > t \implies f(s, x, \xi) \geq 0 \tag{3.2}$$

for any t, $s \in \mathbf{R}$ and $(x, \xi) \in T^*\mathbf{R}^n$. This means that the adjoint P_0^* satisfies condition (Ψ) for the eigenvalue $\tau - if(t, x, \xi)$. Observe that if $\chi \geq 0$ then χf also satisfies (3.2), thus the condition can be localized.

Remark 3.1. *We may also consider symbols $f \in L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$, that is, $f(t, x, \xi) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ is bounded in $S^1_{1,0}(\mathbf{R}^n)$ for almost all t. Then we say that P_0 satisfies condition $(\overline{\Psi})$ if for every (x, ξ) condition* (3.2) *holds for almost all s, $t \in \mathbf{R}$.*

Observe that, since $(x, \xi) \mapsto f(t, x, \xi)$ is continuous for almost all t, it suffices to check (3.2) for (x, ξ) in a countable dense subset of $T^*\mathbf{R}^n$. Then we find that f has a representative satisfying (3.2) for any t, s and (x, ξ) after putting $f(t, x, \xi) \equiv 0$ for t in a null set.

In order to prove Theorem 2.7 we shall make a second microlocalization using the specialized symbol classes of the Weyl calculus, and the Weyl quantization of symbols $a \in \mathcal{S}'(T^*\mathbf{R}^n)$ defined by:

$$(a^w u, v) = (2\pi)^{-n} \iint \exp\left(i\langle x - y, \xi\rangle\right) a\left(\tfrac{x+y}{2}, \xi\right) u(y)\overline{v}(x)\, dx dy d\xi \qquad u, v \in \mathcal{S}(\mathbf{R}^n)$$

Observe that $\operatorname{Re} a^w = (\operatorname{Re} a)^w$ is the symmetric part and $i \operatorname{Im} a^w = (i \operatorname{Im} a)^w$ the antisymmetric part of the operator a^w. Also, if $a \in S^m_{1,0}(\mathbf{R}^n)$ then $a^w(x, D_x) = a(x, D_x)$ modulo $\Psi^{m-1}_{1,0}(\mathbf{R}^n)$ by [14, Theorem 18.5.10]. The same holds for $N \times N$ systems of operators.

We recall the definitions of the Weyl calculus: let g_w be a Riemannean metric on $T^*\mathbf{R}^n$, $w = (x, \xi)$, then we say that g is slowly varying if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies

$$1/C \leq g_w / g_{w_0} \leq C$$

that is, $g_w \cong g_{w_0}$. Let σ be the standard symplectic form on $T^*\mathbf{R}^n$, $g^\sigma(w)$ the dual metric of $w \mapsto g(\sigma(w))$ and assume that $g^\sigma(w) \geq g(w)$. We say that g is σ temperate if it is slowly varying and

$$g_w \leq C g_{w_0}(1 + g^\sigma_w(w - w_0))^N \qquad w,\, w_0 \in T^*\mathbf{R}^n$$

A positive real valued function $m(w)$ on $T^*\mathbf{R}^n$ is g continuous if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies $m(w) \cong m(w_0)$. We say that m is σ, g temperate if it is g continuous and

$$m(w) \le Cm(w_0)(1 + g^\sigma_w(w - w_0))^N \qquad w,\, w_0 \in T^*\mathbf{R}^n$$

If m is σ, g temperate, then m is a weight for g and we can define the symbol classes: $a \in S(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and

$$|a|^g_j(w) = \sup_{T_i \ne 0} \frac{|a^{(j)}(w, T_1, \dots, T_j)|}{\prod_1^j g_w(T_i)^{1/2}} \le C_j m(w) \qquad w \in T^*\mathbf{R}^n \qquad j \ge 0 \tag{3.3}$$

which defines the seminorms of $S(m, g)$. Of course, these symbol classes can also defined locally. For matrix valued symbols, we use the matrix norms. If $a \in S(m, g)$ then we say that the corresponding Weyl operator $a^w \in \operatorname{Op} S(m, g)$. For more results on the Weyl calculus, see [14, Section 18.5].

Definition 3.2. Let m be a weight for the metric g. We say that $a \in S^+(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and $|a|^g_j \le C_j m$ for $j \ge 1$.

Observe that if $a \in S^+(m, g)$ then a is a symbol. In fact, since $g \le g^\sigma$ we find by integration that

$$|a(w) - a(w_0)| \le C_1 \sup_{\theta \in [0,1]} m(w_\theta) g_{w_\theta}(w - w_0)^{1/2} \le C_N m(w_0)(1 + g^\sigma_{w_0}(w - w_0))^{N_0}$$

where $w_\theta = \theta w + (1 - \theta)w_0$, which implies that $m + |a|$ is a weight for g. Clearly, $a \in S(m + |a|, g)$, so the operator a^w is well-defined.

Lemma 3.3. *Assume that m_j is a weight for for the σ temperate conformal metrics $g_j = h_j g^\sharp \le g^\sharp = (g^\sharp)^\sigma$ and $a_j \in S^+(m_j, g_j)$, $j = 1,\ 2$. Let $g = (g_1 + g_2)/2$ and $h^2 = \sup g_1/g_2^\sigma = \sup g_2/g_1^\sigma$, then we find that $h^2 = h_1 h_2$ and*

$$a_1^w a_2^w - (a_1 a_2)^w \in \operatorname{Op} S(m_1 m_2 h, g) \tag{3.4}$$

We also obtain the usual expansion of (3.4) *with terms in $S(m_1 m_2 h^k, g)$, $k \ge 1$.*

Observe that by Proposition 18.5.7 and (18.5.14) in [14] we find that g is σ temperate and $g/g^\sigma \le (h_1 + h_2)^2/4 \le 1$.

Proof. As showed after Definition 3.2 we have that $m_j + |a_j|$ is a weight for g_j and $a_j \in S(m_j + |a_j|, g_j)$, $j = 1,\ 2$. Thus

$$a_1^w a_2^w \in \operatorname{Op} S((m_1 + |a_1|)(m_2 + |a_2|), g)$$

is given by Proposition 18.5.5 in [14]. We find that $a_1^w a_2^w - (a_1a_2)^w = a^w$ with

$$a(w) = E(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2})a_1(w_1)a_2(w_2)\big|_{w_1=w_2=w}$$

where $E(z) = (e^z - 1)/z = \int_0^1 e^{\theta z}\, d\theta$. We have that $\sigma(D_{w_1}, D_{w_2})a_1(w_1)a_2(w_2) \in S(M, G)$ where

$$M(w_1, w_2) = m_1(w_1)m_2(w_2)h_1^{1/2}(w_1)h_2^{1/2}(w_2)$$

and $G_{w_1,w_2}(z_1, z_2) = g_{1,w_1}(z_1) + g_{2,w_2}(z_2)$. Now the proof of Theorem 18.5.5 in [14] works also when $\sigma(D_{w_1}, D_{w_2})$ is replaced by $\theta\sigma(D_{w_1}, D_{w_2})$, uniformly in $0 \le \theta \le 1$. By using Proposition 18.5.7 in [14] and integrating over $\theta \in [0, 1]$ we obtain that $a(w)$ has an asymptotic expansion in $S(m_1m_2h^k, g)$, which proves the Lemma. □

Remark 3.4. *The conclusions of Lemma 3.3 also hold if a_1 has values in $\mathcal{L}(B_1, B_2)$ and a_2 has values in B_1 where B_1 and B_2 are Banach spaces (see Section 18.6 in* [14]*).*

For example, if $\{ a_j \}_j \in S(m_1, g_1)$ with values in ℓ^2, and $b_j \in S(m_2, g_2)$ uniformly in j, then $\left\{ a_j^w b_j^w \right\}_j \in \operatorname{Op}(m_1m_2, g)$ with values in ℓ^2. Thus, if $\{ \phi_j \}_j \in S(1, g)$ is a partition of unity so that $\sum_j \phi_j^2 = 1$ and $a \in S(m, g)$, then $\{ \phi_j a \}_j \in S(m, g)$ has values in ℓ^2.

Example 3.5. The standard symbol class $S^\mu_{\varrho,\delta}$ defined by

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C_{\alpha\beta}\langle\xi\rangle^{\mu+\delta|\alpha|-\varrho|\beta|}$$

has σ temperate metric if $0 \le \delta \le \varrho \le 1$ and $\delta < 1$.

In the proof of Theorem 2.7 we shall microlocalize near (x_0, ξ_0) and put $h^{-1} = \langle\xi_0\rangle = 1 + |\xi_0|$. Then after doing a symplectic dilation: $(x, \xi) \mapsto (h^{-1/2}x, h^{1/2}\xi)$, we find that $S^k_{1,0} = S(h^{-k}, hg^\sharp)$ and $S^k_{1/2,1/2} = S(h^{-k}, g^\sharp)$, $k \in \mathbf{R}$, where $g^\sharp = (g^\sharp)^\sigma$ is the Euclidean metric. We shall prove a semiclassical estimate for a microlocal normal form of the operator.

Let $\|u\|$ be the L^2 norm on $\mathbf{R}^{n+1}$, and (u, v) the corresponding sesquilinear inner product. As before, we say that $f \in L^\infty(\mathbf{R}, S(m, g))$ if $f(t, x, \xi)$ is measurable and bounded in $S(m, g)$ for almost all t. The following is the main estimate that we shall prove.

Proposition 3.6. *Assume that*

$$P_0 = (D_t + if^w(t, x, D_x))\operatorname{Id}_N + F_0^w(t, x, D_x)$$

where $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ is real satisfying condition $(\overline{\Psi})$ given by (3.2), and $F_0 \in L^\infty(\mathbf{R}, S(1, hg^\sharp))$ is an $N \times N$ system, here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. Then *there exists* $T_0 > 0$ *and* $N \times N$ *symbols* $b_T(t,x,\xi) \in L^\infty(\mathbf{R}, S(h^{-1/2}, g^\sharp) \bigcap S^+(1,g^\sharp))$ *such that* $\operatorname{Im} b_T \in L^\infty(\mathbf{R}, S(h^{1/2}, g^\sharp))$ *uniformly for* $0 < T \le T_0$, *and*

$$h^{1/2}\left(\|b_T^w u\|^2 + \|u\|^2\right) \le C_0 T \operatorname{Im}(P_0 u, b_T^w u) \tag{3.5}$$

for $u(t,x) \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^n, \mathbf{C}^N)$ *having support where* $|t| \le T$. *The constants* C_0, T_0 *and the seminorms of* b_T *only depend on the seminorms of* f *and* F_0.

Remark 3.7. *It follows from the proof that* $b_T = \widetilde{b}_T E^* E$ *modulo* $S(h^{1/2}, g^\sharp)$ *where* $E \in S(1, hg^\sharp)$ *is an invertible* $N\times N$ *system,* $\widetilde{b}_T$ *is scalar and* $|\widetilde{b}_T| \le CH^{-1/2}$, *here* H *is a weight for* $g^\sharp$ *such that* $h \le H \le 1$, *and* $G = Hg^\sharp$ *is* σ *temperate (see Claim 3.9, Definition 5.3 and Proposition 6.3).*

Observe that it follows from (3.5) and the Cauchy-Schwarz inequality that

$$\|u\| \le CTh^{-1/2}\|P_0 u\|$$

which will give a loss of 3/2 derivatives after microlocalization. Proposition 3.6 will be proved in Section 7.

There are two difficulties present in estimates of the type (3.5). The first is that b_T is not C^∞ in the t variables, therefore one has to be careful not to involve b_T^w in the calculus with symbols in all the variables. We shall avoid this problem by using tensor products of operators and the Cauchy-Schwarz inequality. The second difficulty lies in the fact that we could have $|b_T| \gg h^{1/2}$, so it is not obvious that cut-off errors can be controlled.

Lemma 3.8. *The estimate* (3.5) *can be perturbed with terms in* $L^\infty(\mathbf{R}, S(h^{1/2}, hg^\sharp))$ *in the expansion of* P_0. *Also, it can be microlocalized: if* $\phi(w) \in S(1, hg^\sharp)$ *is real valued and independent of* t, *then we have*

$$\operatorname{Im}(P_0\phi^w u, b_T^w \phi^w u) \le \operatorname{Im}(P_0 u, \phi^w b_T^w \phi^w u) + Ch^{1/2}\|u\|^2 \qquad u(t,x) \in \mathcal{S}(\mathbf{R}^{n+1}, \mathbf{C}^N) \tag{3.6}$$

where $\phi^w b_T^w \phi^w$ *satisfies the same conditions as* b_T^w.

Proof. In the following, we shall say that a system is *real* if it is a real multiple of the identity matrix. It is clear that we may perturb (3.5) with terms in $L^\infty(\mathbf{R}, S(h^{1/2}, g^\sharp))$ in the expansion of P_0 for small enough T. Now, we can also perturb with real terms $r^w \in L^\infty(\mathbf{R}, \operatorname{Op} S(1, hg^\sharp))$. In fact, if $r \in S(1, hg^\sharp)$ is real and $B \in S^+(1, g^\sharp)$ is symmetric modulo $S(h^{1/2}, g^\sharp)$, then

$$|\operatorname{Im}(r^w u, B^w u)| \le |([(\operatorname{Re} B)^w, r^w]u, u)|/2 + |(r^w u, (\operatorname{Im} B)^w u)| \le Ch^{1/2}\|u\|^2 \tag{3.7}$$

In fact, we have $[(\operatorname{Re} B)^w, r^w] \in \operatorname{Op} S(h^{1/2}, g^\sharp)$ by Lemma 3.3.

If $\phi(w) \in S(1, hg^\sharp)$ then $[P_0, \phi^w \operatorname{Id}_N] = \{ f, \phi \}^w \operatorname{Id}_N$ modulo $L^\infty(\mathbf{R}, \operatorname{Op} S(h, hg^\sharp))$ where $\{ f, \phi \} \in L^\infty(\mathbf{R}, S(1, hg^\sharp))$ is real valued. By using (3.7) with $r^w = \{ f, \phi \}^w \operatorname{Id}_N$ and $B^w = b_T^w \phi^w$, we obtain (3.6) since $b_T^w \phi^w \in \operatorname{Op} S^+(1, g^\sharp)$ is symmetric modulo $\operatorname{Op} S(h^{1/2}, g^\sharp)$ for almost all t by Lemma 3.3. Since Lemma 3.3 also gives that $\phi^w b_T^w \phi^w = \phi^w (b_T \phi)^w = (b_T \phi^2)^w$ modulo $L^\infty(\mathbf{R}, \operatorname{Op} S(h, g^\sharp))$ we find that $\phi^w b_T^w \phi^w$ satisfies the same conditions as b_T^w. □

Claim 3.9. When proving the estimate (3.5) we may assume that

$$F_0 = \langle d_w f, R_0 \rangle = \sum_j \partial_{w_j} f R_{0,j} \qquad \text{modulo } L^\infty(\mathbf{R}, S(h, hg^\sharp)) \tag{3.8}$$

where $R_{0,j} \in L^\infty(\mathbf{R}, S(h^{1/2}, hg^\sharp))$ are $N \times N$ systems, $\forall\, j$.

Proof. By conjugation with $(E^{\pm 1})^w \in \operatorname{Op} S(1, hg^\sharp)$ we find that

$$(E^{-1})^w P E^w = (E^{-1})^w E^w (D_t + i f^w) \operatorname{Id}_N + \big(E^{-1}(D_t E + H_f E + F_0 E)\big)^w = \widetilde{P}$$

modulo $L^\infty(\mathbf{R}, S(h, hg^\sharp))$. By solving

$$\begin{cases} D_t E + F_0 E = 0 \\ E\big|_{t=0} = \operatorname{Id}_N \end{cases}$$

we obtain (3.8) for $\widetilde{P}$ with $\langle d_w f, R_0 \rangle = E^{-1} H_f E$. From the calculus we obtain that

$$E^w (E^{-1})^w = 1 = (E^{-1})^w E^w \qquad \text{modulo } \operatorname{Op} S(Th, hg^\sharp)$$

uniformly when $|t| \le T$. Thus, for small enough T we obtain that $(E^{\pm 1})^w$ is invertible in L^2. Since the metric $hg^\sharp$ is trivially strongly σ temperate in the sense of [2, Definition 7.1], we find from [2, Corollary 7.7] that there exists $A \in L^\infty(\mathbf{R}, S(1, hg^\sharp))$ such that $E^w A^w = 1$. Thus, if we prove the estimate (3.5) for $\widetilde{P}$ and substitute $u = A^w v$ we obtain the estimate for P with b_T replaced by $((E^{-1})^w)^* b_T^w A^w$. Since $A = E^{-1}$ modulo $S(h, hg^\sharp)$ we find from Lemma 3.3 as before that the symbol of this multiplier is in $S(h^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$ and that it is symmetric modulo $S(h^{1/2}, g^\sharp)$. □

We shall see from the proof that if F_0 is on the form (3.8) then $b_T = b_T \operatorname{Id}_N$ is real. Thus, in general the symbol of the multiplier will be on the form $b_T (E^{-1})^* E^{-1}$ modulo $S(h^{1/2}, g^\sharp)$ with invertible E and a real scalar b_T. In the following, we shall use the partial Sobolev norms:

$$\|u\|_s = \|\langle D_x \rangle^s u\| \tag{3.9}$$

We shall now prove the estimate we shall use in the proof of Theorem 2.7.

Proposition 3.10. *Assume that*

$$P_0 = (D_t + iF^w(t,x,D_x))\operatorname{Id}_N + F_0^w(t,x,D_x)$$

with $F^w \in L^\infty(\mathbf{R}, \Psi^1_{cl}(\mathbf{R}^n))$ *having real principal symbol* f *satisfying condition* $(\overline{\Psi})$ *given by* (3.2) *and* $F_0 \in L^\infty(\mathbf{R}, \Psi^0_{cl}(\mathbf{R}^n))$ *is an* $N \times N$ *system. Then there exists* $T_0 > 0$ *and* $N \times N$ *symbols* $B_T(t,x,\xi) \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$ *with*

$$\nabla B_T = (\partial_x B_T, |\xi|\partial_\xi B_T) \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$$

and $\operatorname{Im} B_T(t,x,\xi) \in L^\infty(\mathbf{R}, S^0_{1/2,1/2}(\mathbf{R}^n))$ *uniformly for* $0 < T \le T_0$*, such that*

$$\|B_T^w u\|^2_{-1/2} + \|u\|^2 \le C_0(T \operatorname{Im}(P_0 u, B_T^w u) + \|u\|^2_{-1}) \tag{3.10}$$

for $u \in \mathcal{S}(\mathbf{R}^{n+1}, \mathbf{C}^N)$ *having support where* $|t| \le T$*. The constants* T_0*,* C_0 *and the seminorms of* B_T *only depend on the seminorms of* F *and* F_0 *in* $L^\infty(\mathbf{R}, S^1_{cl}(\mathbf{R}^n))$*.*

Since $\nabla B_T \in L^\infty(\mathbf{R}, S^1_{1/2,1/2})$ we find that the commutators of B_T^w with scalar operators in $L^\infty(\mathbf{R}, \Psi^0_{1,0})$ are in $L^\infty(\mathbf{R}, \Psi^0_{1/2,1/2})$. This will make it possible to localize the estimate. The idea to include the first term in (3.10) is due to Lerner [24].

Proof that Proposition 3.6 gives Proposition 3.10. Choose real symbols $\{\phi_j(x,\xi)\}_j$ and $\{\psi_j(x,\xi)\}_j \in S^0_{1,0}(\mathbf{R}^n)$ having values in ℓ^2, such that $\sum_j \phi_j^2 = 1$, $\psi_j\phi_j = \phi_j$ and $\psi_j \ge 0$. We may assume that the supports are small enough so that $\langle\xi\rangle \cong \langle\xi_j\rangle$ in $\operatorname{supp}\psi_j$ for some ξ_j, and that there is a fixed bound on number of overlapping supports. Then, after doing a symplectic dilation

$$(y,\eta) = (x\langle\xi_j\rangle^{1/2}, \xi/\langle\xi_j\rangle^{1/2})$$

we obtain that $S^m_{1,0}(\mathbf{R}^n) = S(h_j^{-m}, h_j g^\sharp)$ and $S^m_{1/2,1/2}(\mathbf{R}^n) = S(h_j^{-m}, g^\sharp)$ in $\operatorname{supp}\psi_j$, $m \in \mathbf{R}$, where $h_j = \langle\xi_j\rangle^{-1} \le 1$ and $g^\sharp(dy,d\eta) = |dy|^2 + |d\eta|^2$ is constant.

By using the calculus in the y variables we find $\phi_j^w P_0 = \phi_j^w P_{0j}$ modulo $\operatorname{Op} S(h_j, h_j g^\sharp)$, where

$$\begin{aligned} P_{0j} &= (D_t + i(\psi_j F)^w(t,y,D_y))\operatorname{Id}_N + (\psi_j F_0)^w(t,y,D_y) \\ &= (D_t + if_j^w(t,y,D_y))\operatorname{Id}_N + F_j^w(t,y,D_y) \end{aligned} \tag{3.11}$$

with $f_j = \psi_j f \in L^\infty(\mathbf{R}, S(h_j^{-1}, h_j g^\sharp))$ satisfying (3.2), and $F_j \in L^\infty(\mathbf{R}, S(1, h_j g^\sharp))$ uniformly in j. Then, by using Proposition 3.6 and Lemma 3.8 for P_{0j} we obtain symbols

$b_{j,T}(t,y,\eta) \in L^\infty(\mathbf{R}, S(h_j^{-1/2}, g^\sharp) \bigcap S^+(1,g^\sharp))$ such that $\operatorname{Im} b_{j,T} \in S(h_j^{1/2}, g^\sharp)$ uniformly for $0 < T \ll 1$, and

$$\|b_{j,T}^w \phi_j^w u\|^2 + \|\phi_j^w u\|^2 \le C_0 T(h_j^{-1/2} \operatorname{Im}\left(P_0 u, \phi_j^w b_{j,T}^w \phi_j^w u\right) + \|u\|^2) \qquad \forall\, j \tag{3.12}$$

for $u(t,y) \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^n, \mathbf{C}^N)$ having support where $|t| \le T$. Here and in the following, the constants are independent of T.

By substituting $\psi_j^w u$ in (3.12) and summing up we obtain

$$\|B_T^w u\|_{-1/2}^2 + \|u\|^2 \le C_0 T(\operatorname{Im}(P_0 u, B_T^w u) + \|u\|^2) + C_1\|u\|_{-1}^2 \tag{3.13}$$

for $u(t,x) \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^n, \mathbf{C}^N)$ having support where $|t| \le T$. Here

$$B_T^w = \sum_j h_j^{-1/2} \psi_j^w \phi_j^w b_{j,T}^w \phi_j^w \psi_j^w = \sum_j B_{j,T}^w \in L^\infty(\mathbf{R}, \Psi^1_{1/2,1/2})$$

so $\operatorname{Im} B_T \in L^\infty(\mathbf{R}, \Psi^0_{1/2,1/2})$. In fact, since $d\psi_j = 0$ on $\operatorname{supp}\phi_j$ we have

$$\left\{\, \phi_j^w [P_{0j}^w, \psi_j^w] \,\right\}_j \in \Psi^{-1}_{1,0}(\mathbf{R}^n)$$

with values in ℓ^2 for almost all t. Also, $\sum_j \phi_j^2 = 1$ so $\sum_j \phi_j^w\phi_j^w = 1$ modulo $\Psi^{-1}(\mathbf{R}^n)$, and by the finite overlap of supports we find that

$$\begin{aligned}(\langle D_x\rangle^{-1/2} B_T^w)^* \langle D_x\rangle^{-1/2} B_T^w &= (B_T^w)^* \langle D_x\rangle^{-1} B_T^w \\ &= \sum_{|j-k|\le K} (B_{j,T}^w)^* \langle D_x\rangle^{-1} B_{k,T}^w \qquad \text{modulo } \Psi^{-2}\end{aligned}$$

for some N, which implies that

$$\|B_T^w u\|_{-1/2}^2 \le C_K \left(\sum_k \|B_{k,T}^w u\|_{-1/2}^2 + \|u\|_{(-1)}^2 \right)$$

We also have that $\langle D_x\rangle^{-1/2} h_j^{-1/2} \psi_j^w \phi_j^w \in \Psi^0(\mathbf{R}^n)$ uniformly, which gives

$$\|B_{k,T}^w u\|_{-1/2} \le C\|b_{k,T}^w \phi_k^w \psi_k^w u\| \qquad \forall\, k$$

We find that $\nabla B_T \in S^1_{1/2,1/2}$ since Lemma 3.3 gives

$$B_T = \sum_j h_j^{-1/2} b_{j,T} \phi_j^2 \in S^1_{1/2,1/2} \qquad \text{modulo } S^0_{1/2,1/2}$$

where $\phi_j \in S(1, h_j g^\sharp)$ and $b_{j,T} \in S^+(1,g^\sharp)$ for almost all t. For small enough T we obtain (3.10) and the corollary. □

4. Proof of Theorem 2.7

In order to prove the theorem, we first need a preparation result so that we can get the system on a normal form.

Proposition 4.1. *Assume that $P \in S^m_{cl}(M)$ is a $N \times N$ system of principal type having constant characteristics near $(x_0, \xi_0) \in T^*M$, then there exist elliptic $N \times N$ systems A and $B \in S^0_{cl}(M)$ such that*

$$A^w P^w B^w = Q^w = \begin{pmatrix} Q^w_{11} & 0 \\ 0 & Q^w_{22} \end{pmatrix} \in \Psi^m_{cl}$$

microlocally near (x_0, ξ_0). We have that $\sigma(Q_{11}) = \lambda \operatorname{Id}_K$ where the section of eigenvalues $\lambda(w) \in C^\infty$ of $P(w)$ is of principal type, and Q^w_{22} is elliptic.

Thus we obtain the system on a block form. Observe that if $K = 0$ then P is elliptic at (x_0, ξ_0). Since P is of principal type we find by the invariance given by (2.2) that Q is of principal type, so λ vanishes of first order on its zeros.

Proof. Since P_m has constant characteristics by the assumptions, we find that the characteristic equation

$$|P_m(w) - \lambda \operatorname{Id}_N| = 0$$

has a unique local solution $\lambda(w) \in C^\infty$ of multiplicity $K > 0$. Since $P_m(w)$ is of principal type, Proposition 2.10 gives that $d\lambda(w_0) \neq 0$ and the geometric multiplicity $\operatorname{Dim} \operatorname{Ker}(P_m(w) - \lambda(w) \operatorname{Id}_N) \equiv K$ in a neighborhood of $w_0 = (x_0, \xi_0)$. Since the dimension is constant, we may choose a C^∞ base for $\operatorname{Ker}(P_m(w) - \lambda(w) \operatorname{Id}_N)$ in a neighborhood of w_0. By orthogonalizing it, extending to a orthonormal C^∞ base for $\mathbf{C}^N$ and using homogeneity we obtain orthogonal homogeneous E such that

$$E^* P_m E = \begin{pmatrix} \lambda(w) \operatorname{Id}_K & P_{12} \\ 0 & P_{22} \end{pmatrix} = \widetilde{P}_m = \sigma((E^w)^* P^w E^w)$$

Clearly $\operatorname{Ker} \widetilde{P}_m = \{ (z_1, \dots, z_N) : \ z_j = 0 \text{ for } j > K \}$ when $\lambda = 0$ and $d\widetilde{P}_m$ is equal to multiplication with $d\lambda$ on $\operatorname{Ker} \widetilde{P}_m$. Since $\widetilde{P}_m$ is of principal type when $\lambda = 0$ we find that $\operatorname{Im} \widetilde{P}_m \bigcap \operatorname{Ker} \widetilde{P}_m = \{ 0 \}$ at w_0, which implies that P_{22} is invertible. In fact, if it was not invertible there would exists $0 \neq z'' \in \mathbf{C}^{N-K}$ so that $P_{22} z'' = 0$, then

$$0 \neq \widetilde{P}_m {}^t(0, z'') = {}^t(P_{12} z'', 0) \in \operatorname{Im} \widetilde{P}_m \bigcap \operatorname{Ker} \widetilde{P}_m$$

giving a contradiction. By multiplying $\widetilde{P}_m$ from left with

$$\begin{pmatrix} \operatorname{Id}_K & -P_{12} P_{22}^{-1} \\ 0 & \operatorname{Id}_{N-K} \end{pmatrix}$$

we obtain $P_{12} \equiv 0$. Thus, we find that

$$A^w P^w B^w = \begin{pmatrix} Q_{11}^w & Q_{12}^w \\ Q_{21}^w & Q_{22}^w \end{pmatrix} \in \Psi_{cl}^1$$

where $\sigma(Q_{11}) = \lambda \,\mathrm{Id}_K$, $|\sigma(Q_{22})| \neq 0$ and Q_{12}, $Q_{21} \in \Psi_{cl}^0$. Choose a microlocal parametrix $B_{22}^w \in \Psi_{cl}^{-m}$ to Q_{22}^w so that $B_{22}^w Q_{22}^w = Q_{22}^w B_{22}^w = \mathrm{Id}_{N-K}$ modulo C^∞ near w_0. By multiplying from the left with

$$\begin{pmatrix} \mathrm{Id}_K & -Q_{12}^w B_{22}^w \\ 0 & \mathrm{Id}_{N-K} \end{pmatrix} \in \Psi_{cl}^0$$

we obtain that $Q_{12} \in S^{-\infty}$. By multiplying from the right with

$$\begin{pmatrix} \mathrm{Id}_K & 0 \\ -B_{22}^w Q_{21}^w & \mathrm{Id}_{N-K} \end{pmatrix} \in \Psi_{cl}^0$$

we obtain $Q_{21} \in S^{-\infty}$. Note that these multiplications do not change the principal symbols of Q_{jj} for $j = 1, 2$, which finishes the proof. □

Proof of Theorem 2.7. Observe that since P satisfies condition (Ψ) we find that the adjoint P^* satisfies condition ($\overline{\Psi}$). By multiplying with an elliptic pseudodifferential operator, we may assume that $m = 1$. Let P^* have the expansion $P_1 + P_0 + \dots$ where $P_1 = \sigma(P^*) \in S^1$, then it is clear that it suffices to consider $w_0 = (x_0, \xi_0) \in |P_1|^{-1}(0)$, otherwise $P^* \in \Psi_{cl}^1(M)$ is elliptic near w_0 so (2.5) holds and P is microlocally solvable. Now P^* is of principal type having constant characteristics so we find by using Proposition 4.1 that

$$P^* = \begin{pmatrix} Q_{11}^w & 0 \\ 0 & Q_{22}^w \end{pmatrix} \in \Psi_{cl}^1$$

microlocally near w_0, where $\sigma(Q_{11}) = \lambda \,\mathrm{Id}_K$ with $\lambda \in C^\infty$ an eigenvalue of $\sigma(P^*)$ of principal type and Q_{22}^w is elliptic. Since Q_{22}^w is elliptic, it is trivially solvable so we only have to investigate the solvability of Q_{11}^w. Now λ is of principal type by the invariance, so if it does not satisfy condition ($\overline{\Psi}$) then the proof of [14, Theorem 26.4.7] can easily be adapted to this case, since the principal part of the operator is a scalar symbol times the identity matrix.

To prove solvability when condition ($\overline{\Psi}$) is satisfied, we shall prove that there exists ϕ and $\psi \in S_{1,0}^0(T^*M)$ such that $\phi = 1$ in a conical neighborhood of (x_0, ξ_0) and for any $T > 0$ there exists a $K \times N$ system $R_T \in S_{1,0}^{1/2}(M)$ with the property that $\mathrm{WF}(R_T^w) \bigcap T_{x_0}^* M = \emptyset$ and

$$\|\phi^w u\| \le C_1 \left(\|\psi^w P^* u\|_{(3/2-1)} + T\|u\| \right) + \|R_T^w u\| + C_0 \|u\|_{(-1)} \qquad u \in C_0^\infty(M, \mathbf{C}^N) \tag{4.1}$$

Here $\|u\|_{(s)}$ is the L^2 Sobolev norm and the constants are independent of T. Then for small enough T we obtain (2.5) and microlocal solvability, since $(x_0, \xi_0) \notin \mathrm{WF}(1-\phi)^w$. In the case the eigenvalue satisfies condition (Ψ) and (2.4) near x_0 we may choose finitely many $\phi_j \in S^0_{1,0}(M)$ such that $\sum \phi_j \geq 1$ near x_0 and $\|\phi_j^w u\|$ can be estimated by the right hand side of (4.1) for some suitable ψ and R_T. By elliptic regularity of $\{\,\phi_j\,\}$ near x_0, we then obtain the estimate (2.5) for small enough T with $x \neq x_0$ in $\mathrm{WF}(R)$.

Observe that in the case when λ satisfies condition (P) we obtain the estimate (4.1) for $P^* = \lambda(x, D_x)\,\mathrm{Id}_N$ with 3/2 replaced with 1 and $C_1 = \mathcal{O}(T)$ from the Beals-Fefferman estimate, see [1]. Since this estimate can be perturbed with terms in Ψ^0_{cl} for small enough T we get the estimate and solvability in this case. A similar argument gives subelliptic estimates if λ satisfies condition $(\overline{\Psi})$ and the bracket condition, see [14, Chapter 27]. This gives Remark 2.8.

It remains to consider the case $P_1 = \lambda\,\mathrm{Id}_N$, where λ satisfies condition $(\overline{\Psi})$. It is clear that by multiplying with an elliptic factor we may assume that $\partial_\xi \operatorname{Re}\lambda(w_0) \neq 0$, in the microlocal case after a conical transformation. Then, we may use Darboux' theorem and the Malgrange preparation theorem to obtain microlocal coordinates $(t, y; \tau, \eta) \in T^*\mathbf{R}^{n+1}$ so that $w_0 = (0,0;0,\eta_0)$, $t = 0$ on $T^*_{x_0}M$ and $\lambda = q(\tau + if)$ in a conical neighborhood of w_0, where $f \in C^\infty(\mathbf{R}, S^1_{1,0})$ is real and homogeneous satisfying condition (3.2), and $0 \neq q \in S^0_{1,0}$, see Theorem 21.3.6 in [14]. By using the Malgrange preparation theorem and homogeneity we find that

$$P_0(t,x;\tau,\xi) = Q_{-1}(t,x;\tau,\xi)(\tau + if(t,x,\xi))\,\mathrm{Id}_N + F_0(t,x,\xi)$$

where Q_{-1} is homogeneous of degree -1 and F_0 is homogeneous of degree 0 in the ξ variables. By conjugation with elliptic Fourier integral operators and using the Malgrange preparation theorem successively on lower order terms, we obtain that

$$P^* = Q^w(D_t\,\mathrm{Id}_N + i\,(\chi F)^w) + R^w \tag{4.2}$$

microlocally in a conical neighborhood Γ of w_0 as in the proof of Theorem 26.4.7′ in [14]. Here we find that $F \in C^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$ has real principal symbol $f\,\mathrm{Id}_N$ satisfying (3.2), $Q \in S^0_{1,0}(\mathbf{R}^{n+1})$ has principal symbol $q\,\mathrm{Id}_N \neq 0$ in Γ and $R \in S^1_{1,0}(\mathbf{R}^{n+1})$ satisfies $\Gamma \bigcap \mathrm{WF}(R^w) = \emptyset$. Also, $\chi(\tau,\eta) \in S^0_{1,0}(\mathbf{R}^{n+1})$ is equal to 1 in Γ and $|\tau| \leq C|\eta|$ in $\operatorname{supp}\chi(\tau,\eta)$. By cutting off in the t variable we may assume that $F \in L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$. Now, we can follow the proof of Theorem 1.4 in [10]. As before, we shall choose ϕ and ψ so that $\phi = 1$ conical neighborhood of w_0, $\psi = 1$ on $\operatorname{supp}\phi$ and $\operatorname{supp}\psi \subset \Gamma$. Also, we shall choose

$$\phi(t, y; \tau, \eta) = \chi_0(t, \tau, \eta)\phi_0(y, \eta)$$

where $\chi_0(t, \tau, \eta) \in S^0_{1,0}(\mathbf{R}^{n+1})$, $\phi_0(y, \eta) \in S^0_{1,0}(\mathbf{R}^n)$, $t \neq 0$ in $\operatorname{supp} \partial_t \chi_0$, $|\tau| \le C|\eta|$ in $\operatorname{supp} \chi_0$ and $|\tau| \cong |\eta|$ in $\operatorname{supp} \partial_{\tau,\eta}\chi_0$.

Since $|\sigma(Q)| \neq 0$ and $R = 0$ on $\operatorname{supp} \psi$ it is no restriction to assume that $Q \equiv \mathrm{Id}_N$ and $R \equiv 0$ when proving the estimate (4.1). Now, by Theorem 18.1.35 in [14] we may compose $C^\infty(\mathbf{R}, \Psi^m_{1,0}(\mathbf{R}^n))$ with operators in $\Psi^k_{1,0}(\mathbf{R}^{n+1})$ having symbols vanishing when $|\tau| \ge c(1 + |\eta|)$, and we obtain the usual asymptotic expansion in $\Psi^{m+k-j}_{1,0}(\mathbf{R}^{n+1})$ for $j \ge 0$. Since $|\tau| \le C|\eta|$ in $\operatorname{supp} \phi$ and $\chi = 1$ on $\operatorname{supp} \psi$, it suffices to prove (4.1) for $P^* = D_t + iF^w$.

By using Proposition 3.10 on $\phi^w u$, we obtain that

$$(4.3) \quad \|B^w_T \phi^w u\|^2_{-1/2} + \|\phi^w u\|^2 \\ \le C_0 T \left(\operatorname{Im}\left(\phi^w P^* u, B^w_T \phi^w u\right) + \operatorname{Im}\left([P^*, \phi^w \mathrm{Id}_N]u, B^w_T \phi^w u\right)\right) + C_1 \|\phi^w u\|^2_{-1}$$

where $B^w_T \in L^\infty(\mathbf{R}, \Psi^1_{1/2,1/2}(\mathbf{R}^n))$ is an $N \times N$ system with $\nabla B_T \in L^\infty(\mathbf{R}, S^1_{1/2,1/2}(\mathbf{R}^n))$, and $\|u\|_s = \|\langle D_y\rangle^s u\|$ is the partial Sobolev norm in the y variables. Since $|\tau| \le C|\xi|$ in $\operatorname{supp} \phi$ we find that $\|\phi^w u\|_{-1} \le C\|u\|_{(-1)}$ For any $u,\ v \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$ we have that

$$(4.4) \qquad |\,(v, B^w_T u)\,| = |\,\big(\langle D_y\rangle^{1/2} v, \langle D_y\rangle^{-1/2} B^w_T u\big)\,| \le C(\|v\|^2_{1/2} + \|B^w_T u\|^2_{-1/2})$$

where $\langle D_y\rangle = 1 + |D_y|$. Now $\phi^w = \phi^w \psi^w$ modulo $\Psi^{-2}_{1,0}(\mathbf{R}^{n+1})$, thus we find from (4.4) that

$$(4.5) \qquad |\,(\phi^w P^* u, B^w_T \phi^w u)\,| \le C(\|\psi^w P^* u\|^2_{1/2} + \|u\|^2 + \|B^w_T \phi^w u\|^2_{-1/2})$$

where the last term can be cancelled for small enough T in (4.3). We also have to estimate the commutator term $\operatorname{Im}\left([P^*, \phi^w \mathrm{Id}_N]u, B^w_T \phi^w u\right)$ in (4.3). We find

$$[P^*, \phi^w \mathrm{Id}_N] = -(i\partial_t \phi^w - \{\, f, \phi \,\}^w)\,\mathrm{Id}_N \in \Psi^0_{1,0}(\mathbf{R}^{n+1})$$

modulo $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$ by the expansion, where the error term can be estimated by (4.4). Since $\phi = \chi_0 \phi_0$ we find that $\{\, f, \phi \,\} = \phi_0 \{\, f, \chi_0 \,\} + \chi_0 \{\, f, \phi_0 \,\}$, where $\phi_0 \{\, f, \chi_0 \,\} = R_0 \in S^0_{1,0}(\mathbf{R}^{n+1})$ is supported when $|\tau| \cong |\eta|$ and $\psi = 1$. Now $(\tau + if)^{-1} \in S^{-1}_{1,0}(\mathbf{R}^{n+1})$ when $|\tau| \cong |\eta|$, thus by [14, Theorem 18.1.35] we find that $R^w_0 = A^w_1 \psi^w P^*$ modulo $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$ where $A_1 = R_0(\tau + if)^{-1} \in S^{-1}_{1,0}(\mathbf{R}^{n+1})$. As before, we find from (4.4) that

$$(4.6) \quad |\,(R^w_0 u, B^w_T \phi^w u)\,| \le C(\|R^w_0 u\|^2_{1/2} + \|B^w_T \phi^w u\|^2_{-1/2}) \\ \le C_0(\|\psi^w P^* u\|^2_{-1/2} + \|B^w_T \phi^w u\|^2_{-1/2} + \|u\|^2_{-1/2})$$

and also

$$|\,(\partial_t\phi^w u, B_T^w\phi^w u)\,| \le \|R_1^w u\|^2 + \|B_T^w\phi^w u\|_{-1/2}^2$$

where $R_1^w = \langle D_y\rangle^{1/2}\partial_t\phi^w \in \Psi_{1,0}^{1/2}(\mathbf{R}^{n+1})$, thus $t \ne 0$ in $\mathrm{WF}(R_1^w)$.

It only remains to estimate the term $\mathrm{Im}\,((\{\,f,\phi_0\,\}\,\chi_0)^w u, B_T^w\phi^w u)$. Here $(\{\,f,\phi_0\,\}\,\chi_0)^w = \{\,f,\phi_0\,\}^w\,\chi_0^w$ and $\phi^w = \phi_0^w\chi_0^w$ modulo $\Psi_{1,0}^{-1}(\mathbf{R}^{n+1})$. As in (4.4) we find

$$|\,(R^w u, B_T^w v)\,| = |\,\big(\langle D_y\rangle R^w u, \langle D_y\rangle^{-1}B_T^w v\big)\,| \le C(\|u\|^2 + \|v\|^2)$$

for $R \in S_{1,0}^{-1}(\mathbf{R}^{n+1})$, thus we find

$$|\,\mathrm{Im}\,((\{\,f,\phi_0\,\}\,\chi_0)^w u, B_T^w\phi^w u)\,| \le |\,\mathrm{Im}\,(\{\,f,\phi_0\,\}^w\,\chi_0^w u, B_T^w\phi_0^w\chi_0^w u)\,| + C\|u\|^2.$$

The calculus gives $B_T^w\phi_0^w = (B_T\phi_0)^w$ and

$$2i\,\mathrm{Im}\,((B_T\phi_0)^w\,\{\,f,\phi_0\,\}^w) = \{\,B_T\phi_0, \{\,f,\phi_0\,\}\,\}^w = 0$$

modulo $L^\infty(\mathbf{R},\Psi_{1/2,1/2}^0(\mathbf{R}^n))$ since $\nabla(B_T\phi_0) \in L^\infty(\mathbf{R}, S_{1/2,1/2}^1(\mathbf{R}^n))$ and $\{\,f,\phi_0\,\}$ is real. Thus, we obtain

$$|\,\mathrm{Im}\,(\{\,f,\phi_0\,\}^w\,\chi_0^w u, B_T^w\phi_0^w\chi_0^w u)\,| \le C\|\chi_0^w u\|^2 \le C'\|u\|^2 \tag{4.7}$$

and the estimate (4.1) for small enough T, which completes the proof of Theorem 2.7. □

5. The symbol classes and weights

In this section we shall define the symbol classes we shall use. Assume that $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ is scalar and satisfies (3.2), here $0 < h \le 1$ and $g^\sharp = (g^\sharp)^\sigma$ are constant. It is no restriction to change h so that $|f|_1^{g^\sharp} \le h^{-1/2}$, which we assume in what follows. The results shall be uniform in the usual sense, they will only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$. Let

$$X_+(t) = \{\,w \in T^*\mathbf{R}^n : \exists\, s \le t,\ f(s,w) > 0\,\} \tag{5.1}$$

$$X_-(t) = \{\,w \in T^*\mathbf{R}^n : \exists\, s \ge t,\ f(s,w) < 0\,\}. \tag{5.2}$$

Clearly, $X_\pm(t)$ are open in $T^*\mathbf{R}^n$, $X_+(s) \subseteq X_+(t)$ and $X_-(s) \supseteq X_-(t)$ when $s \le t$. By condition $(\overline{\Psi})$ we obtain that $X_-(t)\bigcap X_+(t) = \emptyset$ and $\pm f(t,w) \ge 0$ when $w \in X_\pm(t)$, $\forall\, t$. Let $X_0(t) = T^*\mathbf{R}^n \setminus (X_+(t)\bigcup X_-(t))$ which is closed in $T^*\mathbf{R}^n$. By the definition of $X_\pm(t)$ we have $f(t,w) = 0$ when $w \in X_0(t)$. Let

$$d_0(t_0,w_0) = \inf\,\{\,g^\sharp(w_0 - z)^{1/2} :\ z \in X_0(t_0)\,\} \tag{5.3}$$

be is the $g^\sharp$ distance in $T^*\mathbf{R}^n$ to $X_0(t_0)$ for fixed t_0, it is equal to $+\infty$ in the case that $X_0(t_0) = \emptyset$. By taking the infimum over z we find that $w \mapsto d_0(t,w)$ is Lipschitz continuous with respect to $g^\sharp$ for fixed t when $d_0 < \infty$, i.e.,

$$\sup_{w \neq z \in T^*\mathbf{R}^n} |\delta_0(t,w) - \delta_0(t,z)|/g^\sharp(w-z)^{1/2} \le 1$$

Definition 5.1. We define the signed distance function $\delta_0(t,w)$ by

$$\delta_0 = \operatorname{sgn}(f)\min(d_0, h^{-1/2}) \tag{5.4}$$

where d_0 is given by (5.3) and

$$\operatorname{sgn}(f)(t,w) = \begin{cases} \pm 1, & w \in X_\pm(t) \\ 0, & w \in X_0(t) \end{cases} \tag{5.5}$$

so that $\operatorname{sgn}(f)f \ge 0$.

Remark 5.2. *The signed distance function $w \mapsto \delta_0(t,w)$ given by Definition 5.1 is Lipschitz continuous with respect to the metric $g^\sharp$ with Lipschitz constant equal to 1, $\forall\, t$. We also find that $t \mapsto \delta_0(t,w)$ is non-decreasing, $\delta_0 f \ge 0$, $|\delta_0| \le h^{-1/2}$ and when $|\delta_0| < h^{-1/2}$ we find that $|\delta_0| = d_0$ is given by* (5.3).

In fact, it suffices to show the Lipschitz continuity of $w \mapsto \delta_0(t,w)$ on $\complement X_0(t)$, and then it follows from the Lipschitz continuity of $w \mapsto d_0(t,w)$ when $d_0 < \infty$. Clearly $\delta_0 f \ge 0$, and since $X_+(t)$ is non-decreasing and $X_-(t)$ is non-increasing when t increases, we find that $t \mapsto \delta_0(t,w)$ is non-decreasing.

In the following, we shall treat t as a parameter which we shall suppress, and we shall denote $f' = \partial_w f$ and $f'' = \partial_w^2 f$. We shall also in the following assume that we have choosen $g^\sharp$ orthonormal coordinates so that $g^\sharp(w) = |w|^2$ and $|f'| \le h^{-1/2}$.

Definition 5.3. Let

$$H^{-1/2} = 1 + |\delta_0| + \frac{|f'|}{|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}} \tag{5.6}$$

and $G = Hg^\sharp$.

Observe that $\langle \delta_0 \rangle = 1 + |\delta_0| \le H^{-1/2}$ and

$$1 \le H^{-1/2} \le 1 + |\delta_0| + h^{-1/4}|f'|^{1/2} \le 3h^{-1/2} \tag{5.7}$$

since $|f'| \le h^{-1/2}$ and $|\delta_0| \le h^{-1/2}$. This gives that $hg^\sharp \le 3G$.

Definition 5.4. Let

$$M = |f| + |f'|H^{-1/2} + |f''|H^{-1} + h^{1/2}H^{-3/2} \tag{5.8}$$

then we have that $h^{1/2} \le M \le C_3 h^{-1}$.

The metric G and weight M have the following properties according to Proposition 3.7 in [8].

Proposition 5.5. *We find that $H^{-1/2}$ is Lipschitz continuous, G is σ temperate such that $G = H^2 G^\sigma$ and*

$$H(w) \le C_0 H(w_0)(1 + G_{w_0}(w - w_0)) \tag{5.9}$$

We have that M is a weight for G such that $M \le CH^{-1}$, $f \in S(M, G)$ and

$$M(w) \le C_1 M(w_0)(1 + G_{w_0}(w - w_0))^{3/2} \tag{5.10}$$

Since $G \le g^\sharp \le G^\sigma$ we find that the conditions (5.9) and (5.10) are stronger than the property of being σ temperate (in fact, strongly σ temperate in the sense of [2, Definition 7.1]). Note that $f \in S(M, Hg^\sharp)$ for any choice of $H \ge h$ in Definition 5.4. The following property of G is the most important for the proof.

Proposition 5.6. *Let $H^{-1/2}$ be given by Definition 5.3 for $f \in S(h^{-1}, hg^\sharp)$. There exists $\kappa_1 > 0$ so that if $\langle \delta_0 \rangle = 1 + |\delta_0| \le \kappa_1 H^{-1/2}$ then*

$$f = \alpha_0 \delta_0 \tag{5.11}$$

where $\kappa_1 M H^{1/2} \le \alpha_0 \in S(MH^{1/2}, G)$, which implies that $\delta_0 = f/\alpha_0 \in S(H^{-1/2}, G)$.

This follows directly from Proposition 3.9 in [8]. Next, we shall define the weight m we shall use.

Definition 5.7. For $(t, w) \in \mathbf{R} \times T^*\mathbf{R}^n$ we let

$$m(t, w) = \inf_{t_1 \le t \le t_2} \big\{ |\delta_0(t_1, w) - \delta_0(t_2, w)| + \max\big(H^{1/2}(t_1, w)\langle \delta_0(t_1, w)\rangle^2, H^{1/2}(t_2, w)\langle \delta_0(t_2, w)\rangle^2\big)/2 \big\} \tag{5.12}$$

where $\langle \delta_0 \rangle = 1 + |\delta_0|$.

This weight essentially measures how much $t \mapsto \delta_0(t, w)$ changes between the minima of $t \mapsto H^{1/2}(t, w)\langle \delta_0(t, w)\rangle^2$, which will give restrictions on the sign changes of the symbol. When $t \mapsto \delta_0(t, w)$ is constant for fixed w, we find that $t \mapsto m(t, w)$ is equal to the largest quasi-convex minorant of $t \mapsto H^{1/2}(t,w)\langle\delta_0(t,w)\rangle^2/2$, i.e., $\sup_I m = \sup_{\partial I} m$ for compact intervals $I \subset \mathbf{R}$, see [15, Definition 1.6.3].

The main difference between this weight and the weight in [8] is the use of $H^{1/2}\langle\delta_0\rangle^2$ in the definition of m instead of $H^{1/2}\langle\delta_0\rangle$, and this is due to Lerner [24]. The weight has the following properties according to Propositions 4.3 and 4.4 in [10].

Proposition 5.8. *We have that* $m \in L^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$, $w \mapsto m(t,w)$ *is uniformly Lipschitz continuous,* $\forall\, t$, *and*

$$h^{1/2}\langle\delta_0\rangle^2/6 \le m \le H^{1/2}\langle\delta_0\rangle^2/2 \le \langle\delta_0\rangle/2 \tag{5.13}$$

There exists $C > 0$ *so that*

$$m(t_0,w) \le Cm(t_0,w_0)(1 + |w-w_0|/\langle\delta_0(t_0,w_0)\rangle)^3 \tag{5.14}$$

thus m *is a weight for* $g^\sharp$.

The following result will be essential for the proof of Proposition 3.6 in Section 7, it follows from Proposition 4.5 in [10].

Proposition 5.9. *Let the weight* M *be given by Definition 5.4 and* m *by Definition 5.7. Then there exists* $C_0 > 0$ *such that*

$$MH^{3/2}\langle\delta_0\rangle^2 \le C_0 m \tag{5.15}$$

We have the following convexity property of $t \mapsto m(t,w)$, which will be important for the construction of the multiplier.

Proposition 5.10. *Let* m *be given by Definition 5.7. Then*

$$\sup_{t_1\le t\le t_2} m(t,w) \le \delta_0(t_2,w) - \delta_0(t_1,w) + m(t_1,w) + m(t_2,w) \qquad \forall\, w \tag{5.16}$$

Proof. Since $t \mapsto \delta_0(t,w)$ is monotone, we find that

$$\inf_{\pm(t-t_0)\ge 0} \big(|\delta_0(t,w) - \delta_0(t_0,w)| + H^{1/2}(t,w)\langle\delta_0(t,w)\rangle^2/2\big) \le m(t_0,w) \tag{5.17}$$

Let $t \in [t_1,t_2]$, then by using (5.17) for $t_0 = t_1$, t_2, and taking the infima, we obtain that

$$\begin{aligned} m(t,w) \le \inf_{r\le t_1<t_2\le s} \delta_0(s,w) - \delta_0(r,w) + H^{1/2}(s,w)\langle\delta_0(s,w)\rangle^2/2 + H^{1/2}(r,w)\langle\delta_0(r,w)\rangle^2/2 \\ \le \delta_0(t_2,w) - \delta_0(t_1,w) + m(t_1,w) + m(t_2,w) \end{aligned}$$

which gives (5.16) after taking the supremum. ∎

Next, we shall construct the pseudo-sign $B = \delta_0 + \varrho_0$, which we shall use in Proposition 6.3 to construct the multiplier of Proposition 3.6.

Proposition 5.11. *Assume that δ_0 is given by Definition 5.1 and m is given by Definition 5.7. Then for $T > 0$ there exists real valued $\varrho_T(t,w) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ with the property that $w \mapsto \varrho_T(t,w)$ is uniformly Lipschitz continuous, and*

$$|\varrho_T| \le m \tag{5.18}$$

$$T\partial_t(\delta_0 + \varrho_T) \ge m/2 \qquad \text{in } \mathcal{D}'(\mathbf{R}) \tag{5.19}$$

when $|t| < T$.

Proof. (We owe this argument to Lars Hörmander [17].) Let

$$\varrho_T(t,w) = \sup_{-T\le s\le t}\left(\delta_0(s,w) - \delta_0(t,w) + \frac{1}{2T}\int_s^t m(r,w)\,dr - m(s,w)\right) \tag{5.20}$$

for $|t| \le T$, then

$$\delta_0(t,w) + \varrho_T(t,w) = \sup_{-T\le s\le t}\left(\delta_0(s,w) - \frac{1}{2T}\int_0^s m(r,w)\,dr - m(s,w)\right) + \frac{1}{2T}\int_0^t m(r,w)\,dr$$

which immediately gives (5.19) since the supremum is non-decreasing. Since $w \mapsto \delta_0(t,w)$ and $w \mapsto m(t,w)$ are uniformly Lipschitz continuous by Proposition 5.8, we find by taking the supremum that $w \mapsto \varrho_T(t,w)$ is uniformly Lipschitz continuous. We find from Proposition 5.10 that

$$\delta_0(s,w) - \delta_0(t,w) + \frac{1}{2T}\int_s^t m(r,w)\,dr - m(s,w) \le m(t,w) \qquad -T \le s \le t \le T$$

By taking the supremum, we obtain that $-m(t,w) \le \varrho_T(t,w) \le m(t,w)$ when $|t| \le T$, which proves the result. □

6. The Wick quantization

In order to define the multiplier we shall use the Wick quantization, and we shall also define the function spaces that we shall use. As before, we shall assume that $g^\sharp = (g^\sharp)^\sigma$ and the coordinates are chosen so that $g^\sharp(w) = |w|^2$. For $a \in L^\infty(T^*\mathbf{R}^n)$ we define the Wick quantization:

$$a^{Wick}(x,D_x)u(x) = \int_{T^*\mathbf{R}^n} a(y,\eta)\Sigma^w_{y,\eta}(x,D_x)u(x)\,dyd\eta \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{6.1}$$

using the projections $\Sigma^w_{y,\eta}(x, D_x)$ with Weyl symbol

$$\Sigma_{y,\eta}(x, \xi) = \pi^{-n} \exp(-g^\sharp(x - y, \xi - \eta))$$

(see [7, Appendix B] or [20, Section 4]). We find that a^{Wick}: $\mathcal{S}(\mathbf{R}^n) \mapsto \mathcal{S}'(\mathbf{R}^n)$ so that $(a^{Wick})^* = (\overline{a})^{Wick}$,

$$a \ge 0 \implies \big(a^{Wick}(x, D_x)u, u\big) \ge 0 \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{6.2}$$

and $\|a^{Wick}(x, D_x)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le \|a\|_{L^\infty(T^*\mathbf{R}^n)}$, which is the main advantage with the Wick quantization (see [20, Proposition 4.2]). Now if $a_t(x, \xi) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ depends on a parameter t, then we find that

$$\int_{\mathbf{R}} \big(a_t^{Wick} u, u\big)\, \phi(t)\, dt = \big(A_\phi^{Wick} u, u\big) \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{6.3}$$

where $A_\phi(x, \xi) = \int_{\mathbf{R}} a_t(x, \xi)\phi(t)\, dt$. In fact, if $a \in L^1$ then this follows from the Fubini theorem, in general we obtain this by cutting off a_t on large sets in $T^*\mathbf{R}^n$ and using dominated convergence. We obtain from the definition that $a^{Wick} = a_0^w$ where

$$a_0(w) = \pi^{-n} \int_{T^*\mathbf{R}^n} a(z) \exp(-|w - z|^2)\, dz \tag{6.4}$$

is the Gaussian regularization, thus Wick operators with symmetric symbols have symmetric Weyl symbols.

We also have the following result about the composition of Wick operators according to the proofs of Proposition 3.4 in [22] and Lemma A.1.5 in [24].

Remark 6.1. *Let $a(w)$, $b(w) \in L^\infty$, and let m_1, m_2 be bounded weights for $g^\sharp$. If $|a| \le m_1$ and $|b'| = |\partial b| \le m_2$, then*

$$a^{Wick} b^{Wick} = (ab)^{Wick} + r^w \tag{6.5}$$

with $r \in S(m_1 m_2, g^\sharp)$. In the case when a, b are real valued, $|a| \le m_1$ and $|b''| \le m_2$, we obtain that

$$\operatorname{Re}\big(a^{Wick} b^{Wick}\big) = \left(ab - \frac{1}{2} a' \cdot b'\right)^{Wick} + r^w \tag{6.6}$$

with $r \in S(m_1 m_2, g^\sharp)$. Here a' is the distributional derivative of $a \in L^\infty$ and b' is Lipschitz continuous, so the product is well-defined in L^∞.

If $A \in L^\infty(T^*\mathbf{R}^n)$ is an $M \times N$ system, then we can define A^{Wick} by (6.1) on $u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$. These operators have the same properties as the scalar operators, but of course we need that $M = N$ in order for (6.2) to hold.

In the following, we shall assume that $G = Hg^\sharp \le g^\sharp$ is a slowly varying metric satisfying

$$H(w) \le C_0 H(w_0)(1 + |w - w_0|)^{N_0} \tag{6.7}$$

and that m is a weight for G satisfying (6.7) with H replaced by m. This means that G and m are strongly σ temperate in the sense of [2, Definition 7.1]. Recall the symbol class $S^+(1, g^\sharp)$ defined by Definition 3.2.

Proposition 6.2. *Assume that $a \in L^\infty(T^*\mathbf{R}^n)$ is a $N \times N$ system such that $|a| \le Cm$, then $a^{Wick} = a_0^w$ where $a_0 \in S(m, g^\sharp)$ is given by (6.4). If $a \in S(m, G)$ for $G = Hg^\sharp$ then $a_0 = a$ modulo symbols in $S(mH, G)$. If $|a| \le Cm$ and $a = 0$ in a fixed G ball with center w, then $a \in S(mH^N, G)$ near w for any N. If a is Lipschitz continuous then we have $a_0 \in S^+(1, g^\sharp)$. If $a(t, w)$ and $g(t, w) \in L^\infty(\mathbf{R} \times T^*\mathbf{R}^n)$ are $N \times N$ systems and $\partial_t a(t, w) \ge g(t, w)$ in $\mathcal{D}'(\mathbf{R})$ for almost all $w \in T^*\mathbf{R}^n$, then we find $\big(\partial_t(a^{Wick})u, u\big) \ge \big(g^{Wick}u, u\big)$ in $\mathcal{D}'(\mathbf{R})$ for $u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$.*

Observe that the results are uniform in the metrics and weights. By localization we find, for example, that if $|a| \le Cm$ and $a \in S(m, G)$ in a G neighborhood of w_0, then $a_0 = a$ modulo $S(mH, G)$ in a smaller G neighborhood of w_0. These results are well known, but for convenience we give a short proof.

Proof. Since a is measurable satisfying $|a| \le Cm$, where $m(z) \le C_0 m(w)(1 + |z - w|)^{N_0}$ by (6.7), we find that $a^{Wick} = a_0^w$ where $a_0 = \mathcal{O}(m)$ is given by (6.4). By differentiating on the exponential factor, we find $a_0 \in S(m, g^\sharp)$.

If $a = 0$ in a G ball of radius $\varepsilon > 0$ and center at w, then we can write

$$\pi^n a_0(w) = \int_{|z-w| \ge \varepsilon H^{-1/2}(w)} a(z) \exp(-|w - z|^2)\, dz = \mathcal{O}(m(w) H^N(w))$$

for any N even after repeated differentiation. If $a \in S(m, G)$ then Taylor's formula gives

$$a_0(w) = a(w) + \pi^{-n} \int_0^1 \int_{T^*\mathbf{R}^n} (1 - \theta) \langle a''(w + \theta z) z, z\rangle e^{-|z|^2}\, dz d\theta$$

where $a'' \in S(mH, G)$ because $G = Hg^\sharp$. Since $m(w + \theta z) \le C_0 m(w)(1 + |z|)^{N_0}$ and $H(w + \theta z) \le C_0 H(w)(1 + |z|)^{N_0}$ when $|\theta| \le 1$, we find that $a_0(w) = a(w)$ modulo $S(mH, G)$. Now, the Lipschitz continuity of a means that $\partial a \in L^\infty(T^*\mathbf{R}^n)$. Since $\partial a_0(w) = \pi^{-n} \int_{T^*\mathbf{R}^n} \partial a(z) \exp(-|w - z|^2)\, dz$, we obtain that $a_0 \in S^+(1, g^\sharp)$.

For the final claim, we note that $-\int a(t,w)\phi'(t)\,dt \ge \int g(t,w)\phi(t)\,dt$ for all $0 \le \phi \in C_0^\infty(\mathbf{R})$ and almost all $w \in T^*\mathbf{R}^n$, which by (6.2) and (6.3) gives

$$-\int \left(a^{Wick}(t,x,D_x)u,u\right)\phi'(t)\,dt \ge \int \left(g^{Wick}(t,x,D_x)u,u\right)\phi(t)\,dt \qquad 0 \le \phi \in C_0^\infty(\mathbf{R})$$

for $u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$. □

We shall compute the Weyl symbol for the Wick operator $(\delta_0 + \varrho_T)^{Wick}$, where ϱ_T is given by Proposition 5.11. In the following we shall suppress the t variable.

Proposition 6.3. *Let $B = \delta_0 + \varrho_0$, where δ_0 is given by Definition 5.1 and ϱ_0 is real valued and Lipschitz continuous, satisfying $|\varrho_0| \le m$, where $m \le H^{1/2}\langle\delta_0\rangle^2/2 \le \langle\delta_0\rangle/2$ is a weight for $g^\sharp$. Then we find*

$$B^{Wick} = b^w$$

where $b = \delta_1 + \varrho_1 \in S(\langle\delta_0\rangle, g^\sharp)\bigcap S^+(1,g^\sharp)$ is real, $\delta_1 \in S(H^{-1/2}, g^\sharp)\bigcap S^+(1,g^\sharp)$, and $\varrho_1 \in S(m, g^\sharp)\bigcap S^+(1,g^\sharp)$. Also, there exists $\kappa_2 > 0$ so that $\delta_1 = \delta_0$ modulo $S(H^{1/2}, G)$ when $\langle\delta_0\rangle \le \kappa_2 H^{-1/2}$, which gives $b = \delta_0$ modulo $S(H^{1/2}\langle\delta_0\rangle^2, g^\sharp)$. For any $\lambda > 0$ we find that $|\delta_0| \ge \lambda H^{-1/2}$ and $H^{1/2} \le \lambda/3$ imply that $\operatorname{sgn}(B) = \operatorname{sgn}(\delta_0)$ *and $|B| \ge \lambda H^{-1/2}/3$.*

Proof. Let $\delta_0^{Wick} = \delta_1^w$ and $\varrho_0^{Wick} = \varrho_1^w$. Since $|\delta_0| \le \langle\delta_0\rangle \le H^{-1/2}$, $|\varrho_0| \le m \le \langle\delta_0\rangle/2$ and the symbols are real valued, we obtain from Proposition 6.2 that $b \in S(\langle\delta_0\rangle, g^\sharp)$, $\delta_1 \in S(H^{-1/2}, g^\sharp)$ and $\varrho_1 \in S(m, g^\sharp)$ are real valued. Since δ_0 and ϱ_0 are uniformly Lipschitz continuous, we find that δ_1 and $\varrho_1 \in S^+(1, g^\sharp)$ by Proposition 6.2.

If $\langle\delta_0\rangle \le \kappa H^{-1/2}$ at w_0 for sufficiently small $\kappa > 0$, then we find by the Lipschitz continuity of δ_0 and the slow variation of G that $\langle\delta_0\rangle \le C_0\kappa H^{-1/2}$ in a fixed G neighborhood ω_κ of w_0 (depending on κ). For $\kappa \ll 1$ we find $\delta_0 \in S(H^{-1/2}, G)$ in ω_κ by Proposition 5.6, thus $\delta_1 = \delta_0$ modulo $S(H^{1/2}, G)$ near w_0 by Proposition 6.2 after localization.

When $|\delta_0| \ge \lambda H^{-1/2} > 0$ at w_0, then we find that

$$|\varrho_0| \le m \le \langle\delta_0\rangle/2 \le (1 + H^{1/2}/\lambda)|\delta_0|/2$$

We obtain that $|\varrho_0| \le 2|\delta_0|/3$ so $\operatorname{sgn}(B) = \operatorname{sgn}(\delta_0)$ and $|B| \ge |\delta_0|/3 \ge \lambda H^{-1/2}/3$ when $H^{1/2} \le \lambda/3$, which completes the proof. □

Let m be given by Definition 5.7, then m is a weight for $g^\sharp$ according to Proposition 5.8. We are going to use the symbol classes $S(m^k, g^\sharp)$, $k \in \mathbf{R}$.

Definition 6.4. Let $H(m^k, g^\sharp)$, be the Hilbert space given by [2, Definition 4.1] so that

$$u \in H(m^k, g^\sharp) \iff a^w u \in L^2 \qquad \forall\, a \in S(m^k, g^\sharp) \qquad k \in \mathbf{R} \tag{6.8}$$

We let $\|u\|_k$ be the norm of $H(m^k, g^\sharp)$.

This Hilbert space has the following properties: $\mathcal{S}$ is dense in $H(m^k, g^\sharp)$, the dual of $H(m^k, g^\sharp)$ is naturally identified with $H(m^{-k}, g^\sharp)$, and if $u \in H(m^k, g^\sharp)$ then $u = a_0^w v$ for some $v \in L^2(\mathbf{R}^n)$ and $a_0 \in S(m^{-k}, g^\sharp)$ (see [2, Corollary 6.7]). It follows that $a^w \in \operatorname{Op} S(m^k, g^\sharp)$ is bounded:

$$u \in H(m^j, g^\sharp) \mapsto a^w u \in H(m^{j-k}, g^\sharp) \tag{6.9}$$

with bound only depending on the seminorms of a.

We recall Proposition 6.5 in [10], which shows that the topology in $H(m^{1/2}, g^\sharp)$ can be defined by the operator m^{Wick}.

Proposition 6.5. *Let $B = \delta_0 + \varrho_0$, where δ_0 is given by Definition 5.1 and $|\varrho_0| \le m$. Then there exist positive constants c_1, c_2 and C_0 such that*

$$c_1 h^{1/2}(\|B^{Wick}u\|^2 + \|u\|^2) \le c_2\|u\|_{1/2}^2 \le \big(m^{Wick}u, u\big) \le C_0\|u\|_{1/2}^2 \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{6.10}$$

The constants only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$.

In the following, we let $\|u(t)\|$ be the L^2 norm of $x \mapsto u(t,x) \in \mathbf{C}^N$ in $\mathbf{R}^n$ for fixed t, and $(u(t), v(t))$ the corresponding sesquilinear inner product. Let $\mathcal{B} = \mathcal{B}(L^2(\mathbf{R}^n), \mathbf{C}^N)$ be the set of bounded operators $L^2(\mathbf{R}^n, \mathbf{C}^N) \mapsto L^2(\mathbf{R}^n, \mathbf{C}^N)$. We shall use operators which depend measurably on t in the following sense.

Definition 6.6. We say that $t \mapsto A(t)$ is weakly measurable if $A(t) \in \mathcal{B}$ for all t and $t \mapsto A(t)u$ is weakly measurable for every $u \in L^2(\mathbf{R}^n, \mathbf{C}^N)$, i.e., $t \mapsto (A(t)u, v)$ is measurable for any u, $v \in L^2(\mathbf{R}^n, \mathbf{C}^N)$. We say that $A(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$ if $t \mapsto A(t)$ is weakly measurable and locally bounded in $\mathcal{B}$.

If $A(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$, then we find that the function $t \mapsto (A(t)u, v) \in L^\infty_{loc}(\mathbf{R})$ has weak derivative $\frac{d}{dt}(Au, v) \in \mathcal{D}'(\mathbf{R})$ for any u, $v \in L^2(\mathbf{R}^n, \mathbf{C}^N)$, given by

$$\tfrac{d}{dt}(Au, v)(\phi) = -\int (A(t)u, v)\,\phi'(t)\,dt \qquad \phi(t) \in C_0^\infty(\mathbf{R})$$

If $u(t)$, $v(t) \in L^\infty_{loc}(\mathbf{R}, L^2(\mathbf{R}^n, \mathbf{C}^N))$ and $A(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$, then $t \mapsto (A(t)u(t), v(t)) \in L^\infty_{loc}(\mathbf{R})$ is measurable. We shall use the following multiplier estimate from [8].

Proposition 6.7. *Let $P = D_t + iF(t)$ with $F(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$. Assume that $B(t) = B^*(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$, such that*

$$\tfrac{d}{dt}(Bu, u) + 2\operatorname{Re}(Bu, Fu) \ge (Mu, u) \quad \text{in } \mathcal{D}'(I) \qquad \forall\, u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N) \tag{6.11}$$

where $M(t) = M^*(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$ *and* $I \subseteq \mathbf{R}$ *is open. Then we have*

$$\int (Mu, u)\, dt \le 2 \int \operatorname{Im}(Pu, Bu)\, dt \tag{6.12}$$

for $u \in C_0^1(I, \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N))$.

Proof. Since $B(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$, we may for u, $v \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$ define the regularization

$$(B_\varepsilon(t)u, v) = \varepsilon^{-1} \int (B(s)u, v)\, \phi((t-s)/\varepsilon)\, ds = (Bu, v)\,(\phi_{\varepsilon,t}) \qquad \varepsilon > 0$$

where $\phi_{\varepsilon,t}(s) = \varepsilon^{-1}\phi((t-s)/\varepsilon)$ with $0 \le \phi \in C_0^\infty(\mathbf{R})$ satisfying $\int \phi(t)\, dt = 1$. Then $t \mapsto (B_\varepsilon(t)u, v)$ is in $C^\infty(\mathbf{R})$ with derivative equal to $\frac{d}{dt}\,(Bu, v)\,(\phi_{\varepsilon,t}) = -\,(Bu, v)\,(\phi'_{\varepsilon,t})$. Let I_0 be an open interval such that $I_0 \Subset I$. Then for small enough $\varepsilon > 0$ and $t \in I_0$ we find from condition (6.11) that

$$\tfrac{d}{dt}\,(B_\varepsilon(t)u, u) + 2\operatorname{Re}(Bu, Fu)\,(\phi_{\varepsilon,t}) \ge (Mu, u)\,(\phi_{\varepsilon,t}) \qquad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N) \tag{6.13}$$

In fact, $\phi_{\varepsilon,t} \ge 0$ and $\operatorname{supp}\phi_{\varepsilon,t} \in C_0^\infty(I)$ for small enough ε when $t \in I_0$.

Now for $u(t) \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N))$ and $\varepsilon > 0$ we define

$$B_{\varepsilon,u}(t) = (B_\varepsilon(t)u(t), u(t)) = \varepsilon^{-1} \int (B(s)u(t), u(t))\, \phi((t-s)/\varepsilon)\, ds \tag{6.14}$$

For small enough ε we obtain $B_{\varepsilon,u}(t) \in C_0^1(I_0)$, with derivative

$$\tfrac{d}{dt} B_{\varepsilon,u} = \big((\tfrac{d}{dt} B_\varepsilon)u, u\big) + 2\operatorname{Re}(B_\varepsilon u, \partial_t u)$$

since $B(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$. By integrating with respect to t, we obtain the vanishing average

$$0 = \int \tfrac{d}{dt} B_{\varepsilon,u}(t)\, dt = \int \big((\tfrac{d}{dt} B_\varepsilon)u, u\big)\, dt + \int 2\operatorname{Re}(B_\varepsilon u, \partial_t u)\, dt \tag{6.15}$$

when $u \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N))$. We obtain from (6.13) and (6.15) that

$$0 \ge \iint \big((M(s)u(t), u(t)) + 2\operatorname{Re}(B(s)u(t), \partial_t u(t) - F(s)u(t))\big)\phi((t-s)/\varepsilon)\, ds dt$$

By letting $\varepsilon \to 0$, we find by dominated convergence that

$$0 \ge \int (M(t)u(t), u(t)) + 2\operatorname{Re}(B(t)u(t), \partial_t u(t) - F(t)u(t))\, dt$$

since $u \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N))$ and $M(t)$, $B(t)$, $F(t) \in L^\infty_{loc}(\mathbf{R}, \mathcal{B})$. Here $\partial_t u - Fu = iPu$ and $2\operatorname{Re}(Bu, iPu) = -2\operatorname{Im}(Pu, Bu)$, thus we obtain (6.12) for $u \in C_0^1(I_0, \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N))$. Since I_0 is an arbitrary open subinterval with compact closure in I, this completes the proof of the proposition. □

7. The lower bounds

In this section shall prove Proposition 3.6, which means obtaining lower bounds on

$$2\,\mathrm{Im}\,(P_0u, b_T^wu) = (\partial_t b_T^w u, u) + 2\,\mathrm{Re}\,(F^wu, b_T^wu)$$

where $P_0 = D_t \,\mathrm{Id}_N + iF^w(t, x, D_x)$ with

$$F(t, w) = f(t, w)\,\mathrm{Id}_N + F_0(t, w) \tag{7.1}$$

Here $f \in L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$ is real valued satisfying condition $(\overline{\Psi})$ given by (2.2), $F_0 \in C^\infty(\mathbf{R}, S(1, hg^\sharp))$ and $b_T^w = B_T^{Wick}$ is the symmetric scalar operator given by Proposition 6.3 for this f. Since Proposition 5.11 and Proposition 6.2 give lower bounds on the first term:

$$\partial_t b_T^w = \partial_t B_T^{wick} \ge m^{wick}/2T \qquad \text{in } L^2 \quad |t| \le T$$

it only remains to obtain comparable lower bounds on $\mathrm{Re}\, b_T^w F^w$ by Proposition 6.5.

By Claim 3.9 we may also assume that

$$F_0 = \langle d_w f, R\rangle = \sum_j \partial_{w_j} f R_j \qquad \text{modulo } S(h, hg^\sharp) \qquad \forall\, t \tag{7.2}$$

where $R_j \in S(h^{1/2}, hg^\sharp)$ are $N \times N$ systems, $\forall\, j$. Observe that since $d_w f \in S(MH^{1/2}, G)$, $hg^\sharp \le 3G$ and $h \le MH^{1/2}h^{1/2}$ by (5.8) we find that $F_0 \in S(MH^{1/2}h^{1/2}, G) \subseteq S(1, G)$ and thus $F \in S(M, G)$.

In the following, the results will hold for almost all $|t| \le T$ and will only depend on the seminorms of f in $L^\infty(\mathbf{R}, S(h^{-1}, hg^\sharp))$. We shall suppress the t variable and assume the coordinates chosen so that $g^\sharp(w) = |w|^2$. In order to prove Proposition 3.6 we need to prove the following result.

Proposition 7.1. *Assume that F is given by (7.1)–(7.2) and $B = \delta_0 + \varrho_0$. Here δ_0 is given by Definition 5.1, ϱ_0 is real valued and Lipschitz continuous satisfying $|\varrho_0| \le m$, where $m \le \langle\delta_0\rangle/2$ is given by Definition 5.7. Then we have*

$$\mathrm{Re}\,\big(B^{Wick}F^wu, u\big) \ge (C^wu, u) \qquad \forall\; u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N) \tag{7.3}$$

for some $N \times N$ system $C \in S(m, g^\sharp)$.

Proof of Proposition 3.6. Let $B_T = \delta_0 + \varrho_T$, where $\delta_0 + \varrho_T$ is the pseudo-sign for f given by Proposition 5.11 for $0 < T \le 1$, so that $|\varrho_T| \le m$ and

$$\partial_t(\delta_0 + \varrho_T) \ge m/2T \qquad \text{in } \mathcal{D}'\big(]-T, T[\big) \tag{7.4}$$

If we put $B_T \equiv 0$ when $|t| > T$, then $B_T^{Wick} = b_T^w$ where $b_T(t,w) \in L^\infty(\mathbf{R}, S(H^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp))$ uniformly by Proposition 6.3. We find by Proposition 6.2 and (7.4) that

$$\left((\partial_t B_T)^{Wick} u, u\right) \geq \left(m^{Wick} u, u\right)/2T \qquad \text{in } \mathcal{D}'\left(]-T,T[\right) \tag{7.5}$$

when $u \in \mathcal{S}(\mathbf{R}^n)$. We obtain from Proposition 6.5 that there exist positive constants c_1 and c_2 so that

$$\left(m^{Wick} u, u\right) \geq c_2 \|u\|_{1/2}^2 \geq c_1 h^{1/2} (\|b_T^w u\|^2 + \|u\|^2) \qquad u \in \mathcal{S}(\mathbf{R}^n) \tag{7.6}$$

Here $\|u\|_{1/2}$ is the norm of the Hilbert space $H(m^{1/2}, g^\sharp)$ given by Definition 6.4. By Proposition 7.1, we find for almost all $t \in [-T,T]$ that

$$\operatorname{Re}\left((B_T^{Wick} F^w)\big|_t u, u\right) = (C^w(t) u, u) \quad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N) \tag{7.7}$$

here the $N \times N$ system $C(t) \in S(m, g^\sharp)$ uniformly. We obtain from (6.9), (7.6) and duality that there exists a positive constant c_3 such that

$$|\,(C^w(t)u, u)\,| \leq \|u\|_{1/2} \|C^w(t) u\|_{-1/2} \leq c_3 \|u\|_{1/2}^2 \leq c_3 \left(m^{Wick} u, u\right)/c_2 \tag{7.8}$$

for $u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$ and $|t| \leq T$. We find from (7.5)–(7.8) that

$$(\partial_t b_T^w u, u) + 2 \operatorname{Re}(F^w u, b_T^w u) \geq (1/2T - 2c_3/c_2)\left(m^{Wick} u, u\right) \qquad \text{in } \mathcal{D}'\left(]-T,T[\right)$$

for $u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$. By using Proposition 6.7 with $P = D_t \operatorname{Id}_N + iF^w(t,x,D_x)$, $B = b_T^w$ and $M = m^{Wick}/4T$ we obtain that

$$c_1 h^{1/2} \int \|b_t^w u\|^2 + \|u\|^2 \, dt \leq \int \left(m^{Wick} u, u\right) dt \leq 8T \int \operatorname{Im}(P_0 u, b_T^w u)\, dt$$

if $u \in \mathcal{S}(\mathbf{R} \times \mathbf{R}^n, \mathbf{C}^N)$ has support where $|t| < T \leq c_2/8c_3$. This finishes the proof of Proposition 3.6. □

Proof of Proposition 7.1. First we note that since $B^{Wick} = b^w \in \operatorname{Op} S(\langle \delta_0 \rangle, g^\sharp)$ by Proposition 6.3 and $h^{1/2}\langle \delta_0 \rangle^2 \leq 6m$ by (5.13), we find $B^{Wick} R^w \in \operatorname{Op} S(m, g^\sharp)$ when $R \in S(h^{1/2}, g^\sharp)$. Since $\operatorname{Im} F = \frac{1}{2i}(F - F^*) \in S(1, hg^\sharp)$ we find

$$2 \operatorname{Re}(B^{Wick} i (\operatorname{Im} F)^w) = i[b^w, (\operatorname{Im} F)^w] \in \operatorname{Op} S(h^{1/2}, g^\sharp)$$

thus it suffices to consider symmetric F satisfying (7.2).

We shall localize in $T^*\mathbf{R}^n$ with respect to the metric $G = Hg^\sharp$, and estimate the localized operators. We shall use the neighborhoods

$$\omega_{w_0}(\varepsilon) = \left\{\, w : \ |w - w_0| < \varepsilon H^{-1/2}(w_0) \,\right\} \qquad \text{for } w_0 \in T^*\mathbf{R}^n \tag{7.9}$$

We may in the following assume that ε is small enough so that $w \mapsto H(w)$ and $w \mapsto M(w)$ only vary with a fixed factor in $\omega_{w_0}(\varepsilon)$. Then by the uniform Lipschitz continuity of

$w \mapsto \delta_0(w)$ we can find $\kappa_0 > 0$ with the following property: for $0 < \kappa \le \kappa_0$ there exist positive constants c_κ and ε_κ so that for any $w_0 \in T^*\mathbf{R}^n$ we have

$$|\delta_0(w)| \le \kappa H^{-1/2}(w) \qquad w \in \omega_{w_0}(\varepsilon_\kappa) \qquad \text{or} \tag{7.10}$$

$$|\delta_0(w)| \ge c_\kappa H^{-1/2}(w) \qquad w \in \omega_{w_0}(\varepsilon_\kappa) \tag{7.11}$$

In fact, we have by the Lipschitz continuity that $|\delta_0(w) - \delta_0(w_0)| \le \varepsilon_\kappa H^{-1/2}(w_0)$ when $w \in \omega_{w_0}(\varepsilon_\kappa)$. Thus, if $\varepsilon_\kappa \ll \kappa$ we obtain that (7.10) holds when $|\delta_0(w_0)| \ll \kappa H^{-1/2}(w_0)$ and (7.11) holds when $|\delta_0(w_0)| \ge c\kappa H^{-1/2}(w_0)$.

Let κ_1 be given by Proposition 5.6, κ_2 by Proposition 6.3, and let ε_κ and c_κ be given by (7.10)–(7.11) for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. Using Proposition 6.3 with $\lambda = c_\kappa$ we obtain that $\operatorname{sgn}(B) = \operatorname{sgn}(\delta_0)$ and

$$|B| \ge c_\kappa H^{-1/2}/3 \qquad \text{in } \omega_{w_0}(\varepsilon_\kappa) \tag{7.12}$$

if $H^{1/2} \le c_\kappa/3$ and (7.11) holds in $\omega_{w_0}(\varepsilon_\kappa)$.

Choose real symbols $\{\,\psi_j(w)\,\}_j$ and $\{\,\Psi_j(w)\,\}_j \in S(1,G)$ with values in ℓ^2, such that $\sum_k \psi_j^2 \equiv 1$, $\psi_j\Psi_j = \psi_j$, $\Psi_j = \phi_j^2 \ge 0$ for some $\{\,\phi_j(w)\,\}_j \in S(1,G)$ with values in ℓ^2 so that

$$\operatorname{supp} \phi_j \subseteq \omega_j = \omega_{w_j}(\varepsilon_\kappa)$$

Recall that $B^{Wick} = b^w$ where $b = \delta_1 + \varrho_1$ is given by Proposition 6.3. In particular, $\delta_1 \in S(H^{-1/2}, G)$ when $H^{1/2} \le \kappa_2/2$ and (7.10) holds, since then $\langle \delta_0 \rangle \le \kappa_2 H^{-1/2}$.

Lemma 7.2. *We find that* $A_j = \Psi_j b \operatorname{Re} F \in S(MH^{-1/2}, g^\sharp) \bigcap S^+(M, g^\sharp)$ *uniformly in* j*, and*

$$\operatorname{Re}(b^w F^w) = \sum_j \psi_j^w A_j^w \psi_j^w \qquad \textit{modulo } \operatorname{Op} S(m, g^\sharp) \tag{7.13}$$

We have $A_j^w = \operatorname{Re} b^w F_j^w$ *modulo* $\operatorname{Op} S(m, g^\sharp)$ *uniformly in* j*, where* $F_j = \Psi_j F$.

Proof. Since $b \in S(H^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$, $\psi_j \in S(1,G)$ and $F_j \in S(M,G)$ we obtain that $A_j \in S(MH^{-1/2}, g^\sharp) \bigcap S^+(M, g^\sharp)$ uniformly in j. Proposition 5.9 gives that

$$MH^{3/2}\langle \delta_0 \rangle^2 \le Cm \tag{7.14}$$

thus we may ignore terms in $\operatorname{Op} S(MH^{3/2}\langle \delta_0 \rangle^2, g^\sharp)$. Observe that since $b \in S(H^{-1/2}, g^\sharp)$, $\{\,\psi_k\,\}_k \in S(1,G)$ has values in ℓ^2 and $A_k \in S(MH^{-1/2}, g^\sharp)$ uniformly, Lemma 3.3 and Remark 3.4 gives that the symbols of $b^w F^w$, $b^w F_j^w$ and $\sum_k \psi_k^w A_k^w \psi_k^w$ have expansions in $S(MH^{j/2}, g^\sharp)$. Also observe that in the domains ω_j where $H^{1/2} \ge c > 0$, we find from

Remark 3.4 that the symbols of $\sum_k \psi_k^w A_k^w \psi_k^w$, $b^w F_j^w$ and $b^w F^w$ are in $S(MH^{3/2}, g^\sharp)$ giving the result in this case. Thus, in the following, we shall assume that $H^{1/2} \ll 1$, and we shall consider the neighborhoods where (7.10) or (7.11) holds.

If (7.11) holds then we find that $\langle \delta_0 \rangle \cong H^{-1/2}$ so $S(MH^{1/2}, g^\sharp) \subseteq S(m, g^\sharp)$ in ω_j by (7.14). Since $b \in S^+(1, g^\sharp)$ and $A_j \in S^+(M, g^\sharp)$ we find from Lemma 3.3 and Remark 3.4 that the symbols of both $\operatorname{Re} b^w F^w$ and $\sum_k \psi_k^w A_k^w \psi_k^w$ are equal to $\sum_k \psi_k^2 A_k = \operatorname{Re} bF$ modulo $S(MH^{1/2}, g^\sharp)$ in ω_j. We also find that the symbol of $\operatorname{Re} b^w F_j^w$ is equal to A_j modulo $S(MH^{1/2}, g^\sharp)$, which proves the result in this case.

Next, we consider the case when (7.10) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H^{1/2} \le \kappa_2/2$ in ω_j. Then $\langle \delta_0 \rangle \le \kappa_2 H^{-1/2}$ so $b = \delta_1 + \varrho_1 \in S(H^{-1/2}, G) + S(m, g^\sharp)$ in ω_j by Proposition 6.3. Now b is real and F is symmetric modulo $S(MH, G)$. Thus, by taking the symmetric part of $b^w F^w = \delta_1^w F^w + \varrho_1^w F^w$ we obtain from Lemma 3.3 that the symbol of $\operatorname{Re}(b^w F^w - (bF)^w)$ is in $S(MH^{3/2}, G) + S(MHm, g^\sharp) \subseteq S(m, g^\sharp)$ in ω_j since $M \le CH^{-1}$. Similarly, we find that $A_j^w = \operatorname{Re} b^w F_j^w$ modulo $S(m, g^\sharp)$. Since $A_j \in S(MH^{-1/2}, G) + S(Mm, g^\sharp)$ uniformly, we find that the symbol of $\sum_k \psi_k^w A_k^w \psi_k^w$ is equal to $\operatorname{Re} bF$ modulo $S(m, g^\sharp)$ in ω_j by Remark 3.4, which proves (7.13) and Lemma 7.2. □

Next, we shall show that there exists $N \times N$ system $C_j \in S(m, g^\sharp)$ uniformly, such that

$$\big(A_j^w u, u\big) \ge \big(C_j^w u, u\big) \qquad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N) \tag{7.15}$$

Then we obtain from (7.13) and (7.15) that

$$\operatorname{Re}\,(b^w F^w u, u) \ge \sum_j \big(\psi_j^w C_j^w \psi_j^w u, u\big) + (R^w u, u) \qquad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}^N)$$

where $\sum_j \psi_j^w C_j^w \psi_j^w$ and $R^w \in \operatorname{Op} S(m, g^\sharp)$, which will prove Proposition 7.1.

Thus, it remains to show that there exists $C_j \in S(m, g^\sharp)$ satisfying (7.15). As before we are going to consider the cases when $H^{1/2} \cong 1$ or $H^{1/2} \ll 1$, and when (7.10) or (7.11) holds in $\omega_j = \omega_{w_j}(\varepsilon_\kappa)$ for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. When $H^{1/2} \ge c > 0$ we find that $A_j \in S(MH^{3/2}, g^\sharp) \subseteq S(m, g^\sharp)$ uniformly by (7.14) which gives the lemma with $C_j = A_j$ in this case. Thus, we may assume that

$$H^{1/2} \le \kappa_4 = \min(\kappa_0, \kappa_1, \kappa_2, \kappa_3)/2 \qquad \text{in } \omega_j \tag{7.16}$$

with $\kappa_3 = 2c_\kappa/3$ so that (7.12) follows from (7.11).

First, we consider the case when $H^{1/2} \le \kappa_4$ and (7.11) holds in ω_j. Since $|\delta_0(w)| \ge c_\kappa H^{-1/2}(w)$, we find $\langle \delta_0 \rangle \cong H^{-1/2}$ in ω_j. As before we may ignore terms in $S(MH^{1/2}, g^\sharp) \subseteq$

$S(m, g^\sharp)$ in ω_j by (7.14). Let $f_j = \Psi_j f$, since $\operatorname{sgn}(f) = \operatorname{sgn}(\delta_0) = \operatorname{sgn}(B)$ in ω_j by Proposition 6.3 we find that $f_j B \geq 0$. Since $f_j \in S(M, G)$, we find $f_j^w = f_j^{Wick}$ modulo $\operatorname{Op} S(MH, G)$ by Proposition 6.2, thus we may replace f_j^w with f_j^{Wick}. Since $F_{0,j} \in S(MH^{1/2}h^{1/2}, G)$ by (7.2) we find that $B^{Wick} F_{0,j}^w \in \operatorname{Op} S(MH^{1/2}, g^\sharp)$. Since $|B| \leq CH^{-1/2}$ and $B \in S^+(1, g^\sharp)$, we find from (6.6) in Remark 6.1 that

$$A_j^w = \operatorname{Re} B^{Wick} f_j^{Wick} = (Bf_j)^{Wick} \geq 0 \qquad \text{in } L^2 \text{ modulo } \operatorname{Op} S(MH^{1/2}, g^\sharp)$$

which gives (7.15) in this case.

Finally, we consider the case when (7.10) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H^{1/2} \leq \kappa_4 \leq \kappa$ in ω_j. Then $\langle \delta_0 \rangle \leq 2\kappa H^{-1/2}$ so we obtain from Proposition 5.6 that $\delta_0 \in S(H^{-1/2}, G) \bigcap S(\langle \delta_0 \rangle, g^\sharp)$ in ω_j. We have that $b^w = (\delta_0 + \varrho_0)^{Wick} = B^{Wick}$, where

$$|\varrho_0| \leq m \leq H^{1/2} \langle \delta_0 \rangle^2/2 \leq \langle \delta_0 \rangle/2 \tag{7.17}$$

by Propositions 5.8 and 5.11. Also, we find form Lemma 7.2 that $A_j^w = \operatorname{Re} B^{Wick} F_j^w$ modulo $\operatorname{Op} S(m, g^\sharp)$.

Take $\chi(t) \in C^\infty(\mathbf{R})$ such that $0 \leq \chi(t) \leq 1$, $|t| \geq 2$ in $\operatorname{supp} \chi(t)$ and $\chi(t) = 1$ for $|t| \geq 3$. Let $\chi_0 = \chi(\delta_0)$, then $2 \leq |\delta_0|$ and $\langle \delta_0 \rangle / |\delta_0| \leq 3/2$ in $\operatorname{supp} \chi_0$, thus

$$1 + \chi_0 \varrho_0/\delta_0 \geq 1 - \chi_0 \langle \delta_0 \rangle / 2|\delta_0| \geq 1/4 \tag{7.18}$$

Since $|\delta_0| \leq 3$ in $\operatorname{supp}(1 - \chi_0)$ we find by (7.17) that

$$B = \delta_0 + \chi_0 \varrho_0 = \delta_0(1 + \chi_0 \varrho_0/\delta_0)$$

modulo terms that are $\mathcal{O}(H^{1/2})$. Since $|\delta_0'| \leq 1$ and

$$|\chi_0 \varrho_0/\delta_0| \leq \chi_0 H^{1/2} \langle \delta_0 \rangle^2 / 2|\delta_0| \leq 3H^{1/2} \langle \delta_0 \rangle / 4$$

we find from (6.5) in Remark 6.1 that

$$B^{Wick} = \delta_0^{Wick} B_0^{Wick} \qquad \text{modulo } \operatorname{Op} S(H^{1/2} \langle \delta_0 \rangle, g^\sharp) \tag{7.19}$$

where $B_0 = 1 + \chi_0 \varrho_0/\delta_0 = \mathcal{O}(1)$. Proposition 6.3 gives $(\chi_0 \varrho_0/\delta_0)^{Wick} \in \operatorname{Op} S(H^{1/2} \langle \delta_0 \rangle, g^\sharp)$ and $\delta_0^{Wick} = \delta_1^w$ where $\delta_1 \in S(H^{-1/2}, g^\sharp)$ and $\delta_1 = \delta_0$ modulo $\operatorname{Op} S(H^{1/2}, G)$ in ω_j. Thus Lemma 3.3 and (7.19) gives

$$B^{Wick} = \delta_1^w B_0^{Wick} = \delta_0^w B_0^{Wick} + c^w \qquad \text{modulo } \operatorname{Op} S(H^{1/2} \langle \delta_0 \rangle, g^\sharp) \tag{7.20}$$

where $c \in S(H^{-1/2}, g^\sharp)$ such that $\operatorname{supp} c \bigcap \omega_j = \emptyset$.

We find from Proposition 5.6 that $f = \alpha_0\delta_0$, where $\kappa_1 MH^{1/2} \le \alpha_0 \in S(MH^{1/2}, G)$, so Leibniz' rule gives $\alpha_0^{1/2} \in S(M^{1/2}H^{1/4}, G)$. Let $f_j = \Psi_j f$ and

$$a_j = \alpha_0^{1/2}\phi_j\delta_0 \in S(M^{1/2}H^{-1/4}, G) \bigcap S(M^{1/2}H^{1/4}\langle\delta_0\rangle, g^\sharp) \tag{7.21}$$

Since $\Psi_j = \phi_j^2$ we find $a_j^2 = f_j\delta_0$ and the calculus gives

$$a_j^w(\alpha_0^{1/2}\phi_j)^w = f_j^w \qquad \text{modulo } \operatorname{Op} S(MH, G) \tag{7.22}$$

Since $\operatorname{supp} f_j \bigcap \operatorname{supp} c = \emptyset$ we find that $f_j^w c^w \in \operatorname{Op} S(MH^{3/2}, g^\sharp)$. We also have

$$\operatorname{Re} f_j^w\delta_0^w = a_j^w a_j^w \qquad \text{modulo } \operatorname{Op} S(MH^{3/2}, G) \tag{7.23}$$

and $\operatorname{Im} f_j^w\delta_0^w \in \operatorname{Op} S(MH^{1/2}, G)$. We obtain from (7.20) and (7.22) that

$$f_j^w B^{Wick} = f_j^w(\delta_0^w B_0^{Wick} + c^w + r^w) = f_j^w\delta_0^w B_0^{Wick} + a_j^w r_j^w \quad \text{modulo } \operatorname{Op} S(m, g^\sharp) \tag{7.24}$$

where $r \in S(H^{1/2}\langle\delta_0\rangle, g^\sharp)$ which gives $r_j^w = (\alpha_0^{1/2}\phi_j)^w r^w \in \operatorname{Op} S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. If $\operatorname{Re} A = \frac{1}{2}(A + A^*)$, $\operatorname{Im} A = \frac{1}{2i}(A - A^*)$ and $B^* = B$ then

$$\operatorname{Re}(AB) = \operatorname{Re}(\operatorname{Re} A)B + i[\operatorname{Im} A, B]/2$$

By taking $A = f_j^w\delta_0^w$ and $B = B_0^{Wick}$ we find from (7.23) that

$$\operatorname{Re}\left(f_j^w\delta_0^w B_0^{Wick}\right) = \operatorname{Re}(a_j^w a_j^w B_0^{Wick}) \qquad \text{modulo } \operatorname{Op} S(m, g^\sharp) \tag{7.25}$$

In fact, $B_0 = 1 + \chi_0\varrho_0/\delta_0$ and $(\chi_0\varrho_0/\delta_0)^{Wick} \in \operatorname{Op} S(H^{1/2}\langle\delta_0\rangle, g^\sharp)$ by Proposition 6.2, thus

$$[a^w, B_0^w] = [a^w, (\chi_0\varrho_0/\delta_0)^{Wick}] \in \operatorname{Op} S(MH^{3/2}\langle\delta_0\rangle, g^\sharp)$$

when $a \in S(MH^{1/2}, G)$. Similarly, we find from (7.21) that

$$a_j^w a_j^w B_0^{Wick} = a_j^w(B_0^{Wick} a_j^w + s_j^w) \qquad \text{modulo } \operatorname{Op} S(m, g^\sharp) \tag{7.26}$$

where $s_j = [a_j^w, B_0^{Wick}] \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. Next, we shall use an argument by Lerner [24]. Since $B_0 \ge 1/4$ by (7.18) we find from (7.24)–(7.26) that

$$\operatorname{Re} f_j^w B^{Wick} \ge \frac{1}{4} a_j^w a_j^w + \operatorname{Re} a_j^w S_j^w \qquad \text{in } L^2 \text{ modulo } \operatorname{Op} S(m, g^\sharp) \tag{7.27}$$

where $S_j = r_j + s_j \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. Then by completing the square, we find

$$\operatorname{Re} f_j^w B^{Wick} \ge \frac{1}{4}\left(a_j^w + 2S_j^w\right)^*\left(a_j^w + 2S_j^w\right) \ge 0 \quad \text{in } L^2 \text{ modulo } \operatorname{Op} S(m, g^\sharp) \tag{7.28}$$

since $(S_j^w)^* S_j^w = \overline{S}_j^w S_j^w \in \operatorname{Op} S(MH^{3/2}\langle\delta_0\rangle^2, g^\sharp)$.

But we must also consider $\operatorname{Re} F_{0,j}^w B^{Wick}$, where F_0 satisfies (7.2) so

$$F_{0,j} = \Psi_j F_0 \in S(MH^{1/2}h^{1/2}, G) \tag{7.29}$$

We shall prove that

$$\operatorname{Re} F_{0,j}^w B^{Wick} = \operatorname{Re} a_j^w R_j^w \qquad \text{modulo } \operatorname{Op} S(m, g^\sharp) \tag{7.30}$$

where $R_j \in S(M^{1/2}H^{3/4}, g^\sharp)$, which can then be included in the term given by S_j in (7.27). Since $b = \delta_0 \in S(H^{-1/2}, G)$ modulo $S(H^{1/2}\langle\delta_0\rangle^2, g^\sharp)$ in ω_j by Proposition 6.3 we find

$$\operatorname{Re} F_{0,j}^w B^{Wick} = \operatorname{Re} F_{0,j}^w b^w = (\operatorname{Re} F_{0,j}\delta_0)^w$$

modulo $\operatorname{Op} S(m, g^\sharp)$. We find from (7.21) and (7.29) that $\operatorname{Im} a_j = 0$, so

$$\operatorname{Re} F_{0,j}\delta_0 = \operatorname{Re} \phi_j^2 F_0 \delta_0 = a_j R_j$$

where

$$R_j = \operatorname{Re} \phi_j F_0/\alpha_0^{1/2} \in S(M^{1/2}H^{1/4}h^{1/2}, G) \subseteq S(M^{1/2}H^{3/4}, G)$$

This gives $(\operatorname{Re} F_{0,j}\delta_0)^w = a_j^w R_j^w$ modulo $\operatorname{Op} S(MHh^{1/2}, G) \subseteq \operatorname{Op} S(m, g^\sharp)$, so we obtain (7.30). By adding R_j to S_j in (7.27) and completing the square as in (7.28), we obtain (7.15) in this case. This completes the proof of Proposition 7.1. □

Remark 7.3. *It follows from the proof of Proposition 7.1 that in order to obtain the estimate* (7.3) *it suffices that the lower order term* $F_0 \in S(MH, g^\sharp) \subseteq S(1, g^\sharp)$.

References

1. R. Beals and C. Fefferman, *On local solvability of linear partial differential equations*, Ann. of Math. **97** (1973), 482–498.
2. J.-M. Bony and J.-Y. Chemin, *Espace fonctionnels associés au calcul de Weyl-Hörmander*, Bull. Soc. Math. France **122** (1994), 77–118.
3. F. Colombini, L. Pernazza, and F. Treves, *Solvability and nonsolvability of second-order evolution equations*, Hyperbolic problems and related topics, Grad. Ser. Anal., Int. Press, Somerville, MA, 2003, 111–120.
4. N. Dencker, *On the propagation of singularities for pseudo-differential operators of principal type*, Ark. Mat. **20** (1982), 23–60.
5. ______, *Preparation theorems for matrix valued functions*, Ann. Inst. Fourier (Grenoble) **43** (1993), 865–892.
6. ______, *The solvability of non L^2 solvable operators*, Journees "Equations aux Dérivées Partielles", St. Jean de Monts, France, 1996.
7. ______, *A sufficient condition for solvability*, International Mathematics Research Notices **1999:12** (1999), 627–659.
8. ______, *The resolution of the Nirenberg-Treves conjecture*, Ann. of Math. **163** (2006), 405–444.
9. ______, *The solvability of pseudo-differential operators*, Phase space analysis of partial differential equations, Vol. I, Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa, 2004, 175–200.
10. ______, *On the solvability of pseudo-differential operators*, Sémin. Equ. Dériv. Partielles, Ecole Polytech., Palaiseau, 2005–2006.
11. ______, *The pseudospectrum of systems of semiclassical operators*, Anal. PDE **1** (2008), 323–373.
12. L. Hörmander, *The Weyl calculus of pseudo-differential operators*, Comm. Partial Differential Equations **32** (1979), 359–443.
13. ______, *Pseudo-differential operators of principal type*, Nato Adv. Study Inst. on Sing. in Bound. Value Problems, Reidel Publ. Co., Dordrecht, 1981, 69–96.

14. ______, *The analysis of linear partial differential operators*, vol. I–IV, Springer Verlag, Berlin, 1983–1985.
15. ______, *Notions of convexity*, Birkhäuser, Boston, 1994.
16. ______, *On the solvability of pseudodifferential equations*, Structure of solutions of differential equations (M. Morimoto and T. Kawai, eds.), World Scientific, New Jersey, 1996, 183–213.
17. ______, *The proof of the Nirenberg-Treves conjecture according to N. Dencker och N. Lerner*, Preprint.
18. N. Lerner, *Sufficiency of condition* (Ψ) *for local solvability in two dimensions*, Ann. of Math. **128** (1988), 243–258.
19. ______, *Nonsolvability in* L^2 *for a first order operator satisfying condition* (Ψ), Ann. of Math. **139** (1994), 363–393.
20. ______, *Energy methods via coherent states and advanced pseudo-differential calculus*, Multidimensional complex analysis and partial differential equations (P. D. Cordaro, H. Jacobowitz, and S. Gidikin, eds.), Amer. Math. Soc., Providence, R.I., USA, 1997, 177–201.
21. ______, *Perturbation and energy estimates*, Ann. Sci. Ecole Norm. Sup. **31** (1998), 843–886.
22. ______, *The Wick calculus of pseudo-differential operators and some of its applications.* Cubo Mat. Educ. **5** (2003), 213–236.
23. ______, *Factorization and solvability*, Preprint.
24. ______, *Cutting the loss of derivatives for solvability under condition* (Ψ), Bull. Soc. Math. France **134** (2006), 559-631.
25. H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. **66** (1957), 155–158.
26. G. A. Mendoza and G. Uhlmann, *A necessary condition for local solvability for a class of operators with double characteristics*, J. Funct. Anal. **52** (1983), 252–256.
27. R. D. Moyer, *Local solvability in two dimensions: Necessary conditions for the principal-type case*, Mimeographed manuscript, University of Kansas, 1978.
28. L. Nirenberg and F. Treves, *On local solvability of linear partial differential equations. Part I: Necessary conditions*, Comm. Partial Differential Equations **23** (1970), 1–38, *Part II: Sufficient conditions*, Comm. Pure Appl. Math. **23** (1970), 459–509; *Correction*, Comm. Pure Appl. Math. **24** (1971), 279–288.
29. P. Popivanov and C. Georgiev, *Necessary condition for local solvability of operators with double characteristics*, Annuaire Univ. Sofia Fac. Math. Méc. **75** (1981), 57–71.
30. K. Pravda-Starov, *etude du pseudo-spectre d'opérateurs non auto-adjoints*, Ph.D. thesis, Université de Rennes I, 2006.
31. J.-M. Trépreau, *Sur la résolubilité analytique microlocale des opérateurs pseudodifférentiels de type principal*, Ph.D. thesis, Université de Reims, 1984.

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