Pseudospectra of Semiclassical (Pseudo-) Differential Operators

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1 Introduction

The purpose of this paper is to show how some results from the theory of partial differential equations apply to the study of *pseudospectra* of non-self-adjoint operators, which is a topic of current interest in applied mathematics; see [6, 29].

We will consider operators that arise from the quantization of bounded functions on the phase space $T^*\mathbb{R}^n$. For stronger results in the analytic case, we will assume that our functions are holomorphic and bounded in tubular complex neighborhoods of $T^*\mathbb{R}^n \subset \mathbb{C}^{2n}$.

Let us present the results in a typical example. We consider

$$P(h) = -h^2\Delta + V(x)\,,$$

a semiclassical Schrödinger operator.

We define the *semiclassical pseudospectrum* of the Schrödinger operator $P(h)$ as

$$\Lambda(p) = \overline{\{\xi^2 + V(x) : (x,\xi) \in \mathbb{R}^{2n},\ \operatorname{Im}\langle \xi, V'(x)\rangle \neq 0\}}\,,$$

noting that in the analytic case $\Lambda(p)$ either is empty or is the closure of the set of all values of $p = \xi^2 + V(x)$.

The following result (see Section 3) shows that the resolvent is large inside the pseudospectrum. We first state it in the case of Schrödinger operators satisfying the assumptions above.

THEOREM 1.1 *Suppose that $P(h) = -h^2\Delta + V(x)$, with $V \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then, for any $z \in \{\xi^2 + V(x) : (x,\xi) \in \mathbb{R}^{2n},\ \operatorname{Im}\langle \xi, V'(x)\rangle \neq 0\}$ there exists $u(h) \in L^2(\mathbb{R}^n)$ with the property*

$$\|(P(h) - z)u(h)\| = \mathcal{O}(h^\infty)\|u(h)\|\,. \tag{1.1}$$

In addition, $u(h)$ is localized to a point in phase space (x,ξ) with $p(x,\xi) = z$. More precisely, $WF_h(u) = \{(x,\xi)\}$, where the wave front set $WF_h(u)$ is defined in

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(2.5). *Finally, for every compact $K \Subset \Lambda(p)$, the above result holds uniformly for $z \in K$ in the natural sense. If the potential is real analytic, then we can replace h^∞ by* $\exp(-1/Ch)$.

This result was proven by Davies [5] for Schrödinger operators in one dimension, but as was pointed out in [34], it follows in great generality from a simple adaptation of the now-classical results of Hörmander [12] and Duistermaat and Sjöstrand [8]. The main point is that, unlike in the case of normal operators, the resolvent can be large on open sets as $h \to 0$. That is particularly striking when $P(h)$ has only discrete spectrum.

To guarantee that, we can, for instance, assume that

$$\begin{gathered} \left|\partial_x^\alpha V(x)\right| \leq C_\alpha (1+|x|)^{m-|\alpha|}, \\ \frac{1+|x|^m+|\xi|^2}{C} \leq |\xi^2+V(x)|\,, \quad |(x,\xi)| \geq C\,, \quad \text{where } m>0\,. \end{gathered} \tag{1.2}$$

This is the simplest example of the behavior of the potential: we can make weaker assumptions on V; see the end of Section 3. In the analytic case, we assume that (1.2) holds as $|x| \to \infty$, $|\mathrm{Im}\, x| < c_0$ (and we only need it with $|\alpha| = 0$).

The classical symbol $p = \xi^2 + V(x)$ avoids all sufficiently negative values, and the Fredholm theory guarantees that $P(h)$ has discrete spectrum for h small enough (see Section 2).

We can, in place of the Schrödinger operator $P(h)$, consider the operator with a bounded symbol, $(P(h)-z_1)^{-1}(P(h)-z_2)$, $z_2 \neq z_1$, and this shows that it is sufficient to consider quantization of bounded functions, with all derivatives bounded,

$$p \in \mathcal{C}_{\mathrm{b}}^\infty(T^*\mathbb{R}^n) = \{u \in \mathcal{C}^\infty(T^*\mathbb{R}^n) : \forall \alpha \in \mathbb{N}_0^n \ \partial^\alpha u \in L^\infty(T^*\mathbb{R}^n)\}\,.$$

In that case we give a more general definition of the semiclassical pseudospectrum:

$$\Lambda(p) \stackrel{\text{def}}{=} \overline{p(\{m \in T^*\mathbb{R}^n : \{p,\bar{p}\}(m) \neq 0\})} \subset \Sigma(p) \stackrel{\text{def}}{=} \overline{p(T^*\mathbb{R}^n)}\,, \tag{1.3}$$

where we used the Poisson bracket

$$\{f,g\} = H_f g\,, \quad H_f \stackrel{\text{def}}{=} \sum_{j=1}^n \partial_{\xi_j} f \partial_{x_j} - \partial_{x_j} f \partial_{\xi_j}\,.$$

The nonvanishing of $\{p, \bar{p}\}$ is a classical equivalent of the operator not being normal; see (2.3) and (2.4) below. We note that in the analytic case, we have

$$\Lambda(p) = \emptyset \quad \text{or} \quad \Lambda(p) = \Sigma(p)\,.$$

We also define additional sets

$$\begin{gathered} \Lambda_\pm(p) = \{p(x,\xi) : \pm\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(x,\xi) > 0\} \subset p(T^*\mathbb{R}^n)\,, \\ \Sigma_\infty(p) = \left\{z : \exists (x_j,\xi_j) \to \infty \lim_{j\to\infty} p(x_j,\xi_j) = z\right\}; \end{gathered} \tag{1.4}$$

that is, $\Sigma_\infty(p)$ is the set of limit points of p at infinity.

In the more general setting we can restate our result as follows:

THEOREM 1.2 *Suppose that* $n \geq 2$, $p \in \mathcal{C}_{\mathrm{b}}^{\infty}(T^*\mathbb{R}^n)$, *and* $p^{-1}(z)$ *is compact for a dense set of values* $z \in \mathbb{C}$. *If* $P(h)$ *has the principal part given by* $p^w(x, hD)$, *then*

$$\Lambda(p) \setminus \Sigma_\infty \subset \overline{\Lambda_-(p)}\,,$$

and for every $z \in \Lambda_-(p)$ *there exists* $u(h) \in L^2(\mathbb{R}^n)$ *with the property*

$$\|(P(h) - z)u(h)\| = \mathcal{O}(h^\infty)\|u(h)\|\,. \tag{1.5}$$

In addition, $u(h)$ *is localized to a point in phase space* (x, ξ) *with* $p(x, \xi) = z$. *More precisely,* $WF_h(u) = \{(x, \xi)\}$, *where the wave front set* $WF_h(u)$ *is defined in* (2.5). *Finally, for every compact* $K \Subset \Lambda(p)$, *the above result holds uniformly for* $z \in K$ *in the natural sense.*

If, in addition, p *has a bounded holomorphic continuation to* $\{(x, \xi) \in \mathbb{C}^{2n}, |\mathrm{Im}(x, \xi)| \leq \frac{1}{C}\}$, *then the same conclusions hold with* h^∞ *replaced by* $\exp(-1/Ch)$.

If $n = 1$, *then the same conclusion holds provided that the assumptions of Lemma* 3.2′ *are satisfied.*

We will see in Section 4 that, in general, we cannot construct an almost solution $u(h)$ at an arbitrary interior point z of $\Lambda(p)$. However, at many points $\Lambda(p) \setminus \Lambda_-(p)$ almost solutions can exist; see [23] for explicit examples in dimension 1.

In simple one-dimensional examples, we can already see that the spectrum $\sigma(P(h))$ typically lies deep inside the pseudospectrum $\Lambda(p)$ (the set of values of p in the analytic case); see [4, 5, 29] and Figure 1.1 for a computational manifestation of this phenomenon. Consider, for instance, the following non-self-adjoint operator $P(h) = (hD_x)^2 + i(hD_x) + x^2$: a formal conjugation

$$e^{-\frac{x}{2h}} P(h) e^{\frac{x}{2h}} = (hD_x)^2 + x^2 + \frac{1}{4}$$

shows that the spectrum of $P(h)$ is given by $(2n+1)h + \frac{1}{4}$, while

$$\Lambda(p) = \{z : \mathrm{Re}\, z \geq (\mathrm{Im}\, z)^2\}\,, \quad p = \xi^2 + i\xi + x^2.$$

To see these phenomena for general operators, we need to make assumptions on $z_0 \in \partial\Sigma(p)$. The first one is the principal-type condition,

$$p(x, \xi) = z_0 \Longrightarrow dp(x, \xi) \neq 0\,, \quad m \in T^*\mathbb{R}^n. \tag{1.6}$$

Then we assume a dynamical condition:

(1.7) For some $\lambda \in \mathbb{C}$, *no* complete trajectory of $H_{\mathrm{Re}(\lambda p)}$ is contained in $p^{-1}(z_0)$.

Under these conditions we have the following:

THEOREM 1.3 *Suppose that* $p \in \mathcal{C}_{\mathrm{b}}^{\infty}(T^*\mathbb{R}^n)$ *and the principal part of* $P(h)$ *is given by* $p^w(x, hD)$. *If* $z_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$ *satisfies* (1.6) *and* (1.7), *then for any* $M > 0$ *and for* $h < h_0(M)$, $0 < h_0(M)$,

$$\left\{z : |z - z_0| < Mh \log\left(\frac{1}{h}\right)\right\} \cap \sigma(P(h)) = \varnothing\,.$$

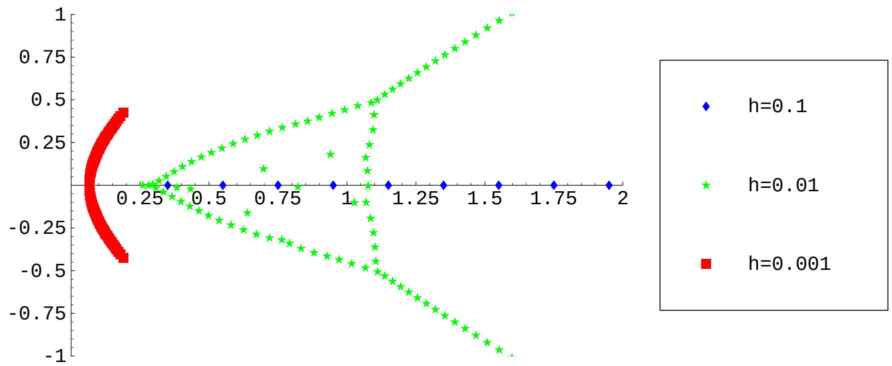

FIGURE 1.1. Numerical computation of the spectrum of $(hD_x)^2 + ihD_x + x^2$. Precise results for larger h, with *false eigenvalues* getting closer to the boundary of the pseudospectrum, $\operatorname{Re} z = (\operatorname{Im} z)^2$, as h decreases.

If, in addition, p is a bounded holomorphic function in a complex tubular neighborhood of $\mathbb{R}^n$, then there exists $C_0 > 0$ such that

$$\left\{ z : |z - z_0| < \frac{1}{C_0} \right\} \cap \sigma(P(h)) = \varnothing \,.$$

As will be explained in a remark in Section 4, we can replace $\Sigma(p)$ in Theorem 1.3 by $\Lambda(p)$. In Section 6 we will show that if (1.7) is violated, then, for a large class of *dissipative operators*, the spectrum lies arbitrarily close (as $h \to 0$) to the boundary of the pseudospectrum. In an example in Section 4, we will indicate why the weaker $\mathcal{C}^\infty$ result is optimal.

Remark. A more complete version of Theorem 1.3, especially in view of Theorem 1.4 below, should also include a bound on the resolvent. The proof in Section 4.2 shows that, under the assumptions of the theorem, $\|(P(h) - z_0)^{-1}\| \leq \frac{C}{h}$, since we can take $\epsilon = Ch$ there.

At the boundary of the pseudospectrum we can expect an improved bound on the resolvent when some additional nondegeneracy is assumed. The result below is based on subelliptic estimates [14, chap. 27], and we borrow our notation from there. If $p = p_1 + ip_2 \in \mathcal{C}^\infty$ with real-valued p_j, then we define the repeated Poisson brackets

$$p_I = H_{p_{i_1}} H_{p_{i_2}} \cdots H_{p_{i_{k-1}}} p_{i_k}$$

where $I = (i_1, i_2, \ldots, i_k) \in \{1, 2\}^k$ and $|I| = k$ is the order of the bracket.

We say that $z_0 \in \Sigma(p) \setminus \Sigma_\infty(p)$ is of *finite type* for p if (1.6) holds at z_0 and for any $(x_0, \xi_0) \in p^{-1}(z_0)$ there exists $k > 1$ and $I \in \{1, 2\}^k$ such that

$$p_I(x_0, \xi_0) \neq 0 \,. \tag{1.8}$$

The *order* of p at $w = (x_0, \xi_0)$ is

$$k(w) = \max\{ j \in \mathbb{Z} : p_I(w) = 0 \text{ for } 1 < |I| \leq j \} \,. \tag{1.9}$$

The order of z_0 is the maximum of the order of p at (x_0, ξ_0) for $(x_0, \xi_0) \in p^{-1}(z_0)$. We say that p satisfies condition (P) if the imaginary part of qp does not change sign on the bicharacteristics of the real part of qp for any $0 \neq q \in \mathcal{C}^\infty$.

As shown in [14, corollary 27.2.4], $k(w) > k$ if and only if

$$\forall z \in \mathbb{C},\ \ j \leq k, \quad (H_{\operatorname{Re} zp})^j \operatorname{Im} zp(w) = 0 \tag{1.10}$$

(see Lemma 5.1 below for a self-contained proof in our special case), and this provides a reformulation of the assumptions of the following:

THEOREM 1.4 *Assume that $p \in \mathcal{C}_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$ and that the principal part of $P(h)$ is $p^w(x, hD)$. If $z_0 \in \partial\Sigma(p)$ is of finite type for p of order $k \geq 1$, then k is* even *and for $h < h_0$, $0 < h_0$,*

$$\|(P(h) - z_0)^{-1}\| \leq C h^{-\frac{k}{k+1}} . \tag{1.11}$$

In particular, there exists $c_0 > 0$ such that

$$\left\{ z : |z - z_0| \leq c_0 h^{\frac{k}{k+1}} \right\} \cap \sigma(P(h)) = \varnothing \,, \quad 0 < h \leq h_0 \,. \tag{1.12}$$

In one dimension this result was proven in [35], and in some special cases by Boulton [2], who also showed that the bounds are optimal. As was demonstrated by Trefethen [30], that is also easy to see numerically.

Remark. After the statement of Theorem 1.3, we pointed out that it can be complemented by a resolvent estimate corresponding to $k = \infty$ in (1.11). Symmetrically, we have a larger region free of spectrum than the one immediately implied by (1.11) and given in (1.12): c_0 should be allowed to be arbitrarily large. This follows from modifying the weight in the proof of Lemma 5.3.

A simple, higher-dimensional example to which Theorem 1.4 applies can be constructed as follows: Let $W \in \mathcal{C}_{\mathrm{b}}^\infty(\mathbb{R}^2)$ be a nonnegative function, vanishing on the circle $x_1^2 + x_2^2 = 1$. Consider

$$P(h) = -h^2 \Delta + i W(x) + i \left(x_1^2 + x_2^2 - 1 \right)^m \quad \text{with } m \text{ even.}$$

Then estimate (1.11) holds for $z_0 > 0$ uniformly on compact subsets of $(0, \infty)$ with $k = 2m$. The increase in k is due to the (simple) tangency of some bicharacteristics of the real part to the set where the imaginary part vanishes.

We conclude by pointing out that we could have defined the semiclassical pseudospectrum of $P(h)$, $\Lambda(P)$, as the closure of the set of points z at which (1.5) holds. We have shown that

$$\Sigma(p) \supset \Lambda(P) \supset \Lambda(p) \,.$$

An equality is not true in general, but we could perhaps hope for

$$\overline{\Lambda(P)^\circ} = \Lambda(p)$$

under suitable assumptions.

Another important topic not explored in this paper is the behavior of evolution operators $\exp(it\frac{P}{h})$ for nonnormal P's and its relation to semiclassical pseudospectra.

We also mention that the interplay between classical properties of symbols and the existence of localized quasi modes can be observed in the Berezin-Toeplitz quantization of compact symplectic Kähler manifolds. For the case of the torus, a very concrete treatment has recently been provided by Chapman and Trefethen [3], who obtained exponential localization for symbols with only partial analytic regularity. The general case in the $\mathcal{C}^\infty$ microlocal framework is presented in [1].

2 Review of Semiclassical Quantization

For simplicity, we will consider the case of semiclassical quantization of functions $p \in \mathcal{C}_\mathrm{b}^\infty(T^*\mathbb{R}^n)$, that is, that p is bounded with bounded derivatives of all orders.

In the analytic case we will assume that $p(x,\xi)$ is bounded and holomorphic in a tubular neighborhood of $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n} \subset \mathbb{C}^{2n}$. As pointed out in the introduction, the case of functions that omit a value in $\mathbb{C}$ and that tend to infinity as $(x,\xi) \to \infty$ can be reduced to this case (see also the remark at the end of Section 3).

We use the Weyl quantization

$$p^w(x, hD_x)u = \frac{1}{(2\pi h)^n} \iint p\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y)dy\,d\xi\,, \tag{2.1}$$

which for $p \in \mathcal{C}_\mathrm{b}^\infty(T^*\mathbb{R}^n)$ gives operators bounded on $L^2(\mathbb{R}^n)$; see [7, chap. 7]. We can consider more general operators

$$P(h) \sim \sum_{j=0}^\infty h^j p_j^w(x, hD)\,,$$

in which case we call p_0 the principal part of $P(h)$.

In the case of analytic symbols we assume that

$$P(x,\xi;h) \sim \sum_{j=0}^\infty h^j p_j(x,\xi)$$

in the space of bounded holomorphic functions in a tubular neighborhood of the real phase space. Although it is not, strictly speaking, necessary for our final conclusions, in the analytic case we make an additional assumption that

$$|p_j(z,\zeta)| \le C^j j^j\,, \quad (z,\zeta) \in \mathbb{C}^n\,, \quad |\mathrm{Im}(z,\zeta)| \le \frac{1}{C}\,. \tag{2.2}$$

That allows us exponentially small errors in the expansions.

The product formula of the Weyl calculus says that

$$p_1^w(x, hD) \circ p_2^w(x, hD) = (p_1 \sharp_h p_2)^w(x, hD; h) \tag{2.3}$$

where

$$(p_1 \# p_1)(x, \xi; h) = e^{\frac{ih}{2}\omega((D_x, D_\xi),(D_y, D_\eta))} p_1(x, \xi) p_2(y, \eta)\big|_{y=x,\, \eta=\xi}$$

has the following asymptotic expansion:

$$p_1 \sharp_h p_2(x, \xi; h) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{ih}{2} \omega((D_x, D_\xi), (D_y, D_\eta)) \right)^k p_1(x, \xi) p_2(y, \eta) \Big|_{y=x, \eta=\xi}, \tag{2.4}$$

with $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$, the symplectic form on $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, and $D_\bullet = (\frac{1}{i})\partial_\bullet$. The expansion determines $p_1 \sharp_h p_2$ up to a term in $\mathcal{O}(h^\infty)\mathcal{C}_{\mathrm{b}}^\infty$. In the analytic case, by summing up to $k \sim 1/(Ch)$, we can obtain $\mathcal{O}(e^{-1/(Ch)})$ errors; see [24].

A basic tool of microlocal analysis is the FBI transform

$$T : L^2(\mathbb{R}^n) \to L^2(T^*\mathbb{R}^n),$$

defined by

$$Tu(x, \xi) = c_n h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{\frac{i}{h}\frac{\langle x-y, \xi\rangle + i|x-y|^2}{2}} u(y) dy.$$

Roughly speaking, its role can be described as follows: the phase space properties of $u \in L^2(\mathbb{R}^n)$ are reflected by the behavior of $Tu \in L^2(T^*\mathbb{R}^n)$ as $h \to 0$. In this note we will deal only with h-dependent smooth functions with a tempered behavior, $|D^\alpha u| = \mathcal{O}(h^{-N_\alpha})$. The notion of the *wave front set* of u, $WF_h(u)$, explains the localization statement in Theorem 1.2 (see also Theorem 1.2′ below). In the $\mathcal{C}^\infty$ case, the h-wave-front set is defined by

$$(x^0, \xi^0) \notin WF_h(u) \iff \forall N \ \ |Tu(x, \xi)| \leq C_N h^{-N} \quad \text{for } (x, \xi) \text{ in a neighborhood of } (x^0, \xi^0), \tag{2.5}$$

and in the analytic case by

$$(x^0, \xi^0) \notin WF_h(u) \iff \exists c > 0 \ \ |Tu(x, \xi)| \leq e^{-\frac{c}{h}} \quad \text{for } (x, \xi) \text{ in a neighborhood of } (x^0, \xi^0). \tag{2.6}$$

In the $\mathcal{C}^\infty$ case we can also characterize $WF_h(u)$ using pseudodifferential operators:

$$(x^0, \xi^0) \notin WF_h(u) \iff \exists p \in \mathcal{C}_{\mathrm{c}}^\infty(T^*\mathbb{R}^n) \ \ p(x^0, \xi^0) \neq 0, \ p(x, hD)u = \mathcal{O}(h^\infty).$$

In the analytic case we will need to understand the action of $P(h)$ on microlocally weighted spaces $H(\Lambda_{tG})$ whose definition, in the simple setting needed here, we now recall; see [10] for the origins of the method and [20] for a recent presentation.

The complexification of the symplectic manifold $T^*\mathbb{R}^n$, $T^*\mathbb{C}^n$, is equipped with the complex symplectic form $\omega_{\mathbb{C}}$ and two natural real symplectic forms $\operatorname{Im}\omega_{\mathbb{C}}$ and $\operatorname{Re}\omega_{\mathbb{C}}$. We see that $T^*\mathbb{R}^n$ is Lagrangian with respect to the first form and symplectic with respect to the second. In general, we call a submanifold satisfying these two conditions an *IR manifold*.

Suppose that $G \in \mathcal{C}_{\rm c}^\infty(T^*\mathbb{R}^n)$. We associate to it a natural family of IR manifolds

$$\Lambda_{tG} = \{\rho + itH_G(\rho) : \rho \in T^*\mathbb{R}^n\} \subset T^*\mathbb{C}^n \quad \text{with } t \in \mathbb{R} \text{ and } |t| \text{ small.} \tag{2.7}$$

Since $\operatorname{Im}(\zeta\, dz)$ is closed on Λ_{tG}, there exists a function H_t on Λ_{tG} such that

$$dH_t = -\operatorname{Im}(\zeta\, dz)\big|_{\Lambda_{tG}} \,,$$

and in fact we can write it down explicitly, parametrizing Λ_{tG} by $T^*\mathbb{R}^n$:

$$H_t(z,\zeta) = -\langle \xi, t\nabla_\xi G(x,\xi)\rangle + tG(x,\xi)\,, \quad (z,\zeta) = (x,\xi) + itH_G(x,\xi)\,.$$

The associated spaces $H(\Lambda_{tG})$ are defined as follows: The FBI transform $Tu(x,\xi)$ is analytic in (x,ξ), and we can continue it to Λ_{tG}. That defines $T_{\Lambda_{tG}}u \in \mathcal{C}^\infty(\Lambda_{tG})$. Since Λ_{tG} differs from $T^*\mathbb{R}^n$ on a compact set only, $T_{\Lambda_{tG}}u$ is square-integrable on Λ_{tG}.

The spaces $H(\Lambda_{tG})$ are defined by putting h-dependent norms on $L^2(\mathbb{R}^n)$:

$$\|u\|^2_{H(\Lambda_{tG})} = \int_{\Lambda_{tG}} |T_{\Lambda_{tG}}u(z,\zeta)|^2 e^{-\frac{2H_t(z,\zeta)}{h}} \frac{(\omega|_{\Lambda_{tG}})^n}{n!}\,.$$

The main result relates the action of a pseudodifferential operator to multiplication by its symbol. Suppose that p_1 and p_2 are bounded and holomorphic in a neighborhood of $T^*\mathbb{R}^n$ in $\mathbb{C}^{2n}$ (see (2.2)). Then for t small enough

$$\begin{aligned} &\left\langle p_1^w(x,hD)u,\, p_2^w(x,hD)v\right\rangle_{H(\Lambda_{tG})} \\ &\quad = \left\langle (p_1|_{\Lambda_{tG}})T_{\Lambda_{tG}}u,\, (p_2|_{\Lambda_{tG}})T_{\Lambda_{tG}}v\right\rangle_{L^2(\Lambda_{tG}, e^{-2H_t/h}(\omega|_{\Lambda_{tG}})^n/n!)} \\ &\qquad + \mathcal{O}(h)\|u\|_{H(\Lambda_{tG})}\|v\|_{H(\Lambda_{tG})}\,; \end{aligned} \tag{2.8}$$

see [10, 20]. In particular, by taking $p_1 = p$, $p_2 = \bar{p}$, and $u = v$, we obtain

$$\begin{aligned} &\|p^w(x,hD)u\|^2_{H(\Lambda_{tG})} = \\ &\qquad \|p|_{\Lambda_{tG}}T_{\Lambda_{tG}}u\|^2_{L^2(\Lambda_{tG}, e^{-2H_t/h}(\omega|_{\Lambda_{tG}})^n/n!)} + \mathcal{O}(h)\|u\|^2_{H(\Lambda_{tG})}\,. \end{aligned} \tag{2.9}$$

For use in the next section, we also recall some basic facts about *positive Lagrangian submanifolds* of a complex symplectic manifold $T^*\mathbb{C}^n$. A complex plane λ of (complex) dimension n is Lagrangian and positive if

$$\forall X, Y \in \lambda \quad \omega_{\mathbb{C}}(X,Y) = 0,\; i\omega_{\mathbb{C}}(\bar{X}, X) \geq 0\,. \tag{2.10}$$

The crucial characterization is given as follows (see [14, prop. 21.5.9]):

(2.11) $\lambda \subset T^*\mathbb{C}^n$ is a positive Lagrangian plane $\Longleftrightarrow \lambda = \{(z, Az) : z \in \mathbb{C}^n\}$

where $A = A_1 + iA_2$ is a symmetric matrix with A_1 real and A_2 positive definite.

3 Semiclassical Pseudospectrum

In Section 1 we defined the semiclassical pseudospectrum $\Lambda(p)$ as the closure of the set of values of p with nonvanishing bracket. We also introduced

$$\Lambda_\pm(p) = \{p(x,\xi) : \pm\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(x,\xi) > 0\} \subset p(T^*\mathbb{R}^n)\,,$$
$$\Sigma_\infty(p) = \Big\{z : \exists (x_j,\xi_j) \to \infty \lim_{j\to\infty} p(x_j,\xi_j) = z\Big\}\,.$$

In the $\mathcal{C}^\infty$ case, Theorem 1.2 follows immediately from a semiclassical reformulation of the nonpropagation of singularities [8, 12, 15]; see [14, sec. 26.3] and [34].

The analytic case is also well-known (see [16]), but since a ready-to-use reference is not available, we include a proof. It can also be adapted to give a self-contained proof in the $\mathcal{C}^\infty$ case.

THEOREM 1.2′ *Suppose that $n \geq 2$, $p(x,\xi)$ satisfies the assumptions in Section* 2 *in the analytic case, and $\Lambda_-(p)$ is given by* (1.4)*. Then*

$$\overline{\Lambda_-(p)} \supset \Lambda(p) \setminus \Sigma_\infty(p)\,,$$

and for every $z \in \Lambda_-(p)$ and every $(x^0,\xi^0) \in T^*\mathbb{R}^n$ with

$$p(x^0,\xi^0) = z\,, \quad \{\mathrm{Re}\, p, \mathrm{Im}\, p\}(x^0,\xi^0) < 0\,,$$

there exists $0 \neq u(h) \in L^2(\mathbb{R}^n)$ such that

(3.1) $$\|(P(h)-z)u(h)\| = \mathcal{O}\big(e^{-\frac{1}{Ch}}\big)\|u(h)\|\,, \quad WF_h(u(h)) = \{(x^0,\xi^0)\}\,.$$

If $n = 1$, then the same conclusion holds provided that the assumptions of Lemma 3.2′ *are satisfied.*

In dimension 1 the theorem holds as well, but further assumptions need to be made on p; see the remark after Lemma 3.2.

Before the proof we want to stress the need for an open dense subset $\Lambda_-(p)$ (it could be a larger subset; see the remark below). One could ask if any interior point of $\Lambda(p) \setminus \Sigma_\infty(p)$ is an "almost eigenvalue" or "quasi mode" in the sense of (3.1). That is not so as shown by the following example.

Example. Consider the following bounded analytic function on $T^*\mathbb{R}$:

$$p(x,\xi) = \frac{\xi^2 - 1 + i\xi x^2(1+x^2)^{-1}}{1 + \xi^2 + i\xi x^2(1+x^2)^{-1}}\,.$$

We see that $p^{-1}(0) = \{(0,1),(0,-1)\}$ and that 0 is an interior point of the pseudospectrum, $0 \in \Lambda(p)^\circ$. Also, 0 is a boundary point of images of neighborhoods of $(0,\pm1)$ under p.

Near $(0, \pm 1)$, p is microlocally equivalent to a nonvanishing multiple of $\xi + i x^2$. An explicit computation shows that the inverse of the models are bounded by $h^{-2/3}$, and a localization argument[1] then shows that

$$\| p^w(x, hD)^{-1} \|_{L^2 \to L^2} \leq h^{-\frac{2}{3}} .$$

Hence, 0 is not a quasi mode.

Remark. We stress that the vanishing of the Poisson bracket $\{\operatorname{Re} p, \operatorname{Im} p\}$ (which occurs in this example at $p^{-1}(0)$) is not enough to guarantee the absence of a quasi mode. As was pointed out by Lerner [17], a violation of the condition $(\overline{\Psi})$ (see [14, sec. 26.4]) can produce quasi modes with the simplest example coming from adapting [14, theorem 26.3.6] as in [34]: $p(x, \xi) = \xi - i x^k$ with $k > 1$ odd. That was done by an explicit construction in [23]. On the other hand, if $z \in \Lambda_+(p)$, $p(m) = z$, and $\{\operatorname{Re} p, \operatorname{Im} p\}(m) > 0$, then $(P(h) - z)u = \mathcal{O}(h^\infty)$ implies $m \notin WF_h(u)$, which is a microhypoellipticity statement.

We start the proof of Theorem 1.2′ with the discussion of $\Lambda_\pm(p)$. To establish that $\Lambda_-(p)$ is dense, we need the following result of Melin and Sjöstrand [22, lemma 8.1]:

LEMMA 3.1 *Suppose that $n \geq 2$ and that $d \operatorname{Re} p$ and $d \operatorname{Im} p$ are linearly independent on $p^{-1}(z)$. If ω is the symplectic form on $T^*\mathbb{R}^n$, then*

$$\{\operatorname{Re} p, \operatorname{Im} p\} \lambda_{p,z} = \left. \frac{\omega^{n-1}}{(n-1)!} \right|_{p^{-1}(z)},$$

where $\lambda_{p,z}$ is the Liouville measure on $p^{-1}(z)$, $\lambda_{p,z} \wedge d \operatorname{Re} p \wedge d \operatorname{Im} p = \omega^n / n!$. In particular, for any compact, connected component of $p^{-1}(z)$, call it Γ, we have

$$\int_\Gamma \{\operatorname{Re} p, \operatorname{Im} p\} \lambda_{p,z}(d\rho) = 0 .$$

As an immediate consequence, we see that $\overline{\Lambda_-(p)} = \Lambda(p)$ if the assumptions of Theorem 1.2 are satisfied, since we have the following:

LEMMA 3.2 *If the assumptions on p are satisfied, $n \geq 2$, and either $\Lambda_+(p)$ or $\Lambda_-(p)$ are nonempty, then in the analytic case $\Lambda_+(p) \cup \Lambda_-(p)$ is dense in $\Sigma(p)$, and in general*

$$\overline{\Lambda_\pm(p)} \supset \Lambda(p) \setminus \Sigma_\infty(p) .$$

PROOF: Assume that $\{\operatorname{Re} p, \operatorname{Im} p\} \not\equiv 0$. Then

$$H \stackrel{\text{def}}{=} \{\rho \in T^*\mathbb{R}^n : \{\operatorname{Re} p, \operatorname{Im} p\}(\rho) = 0\}$$

is an analytic hypersurface without any interior points. Consequently, every value $z = p(\rho)$ with $\rho \in H$ can be approximated by values $z_j = p(\rho_j)$ with $\mathbb{R}^{2n} \setminus H \ni \rho_j \to \rho$, and $\Lambda_+(p) \cup \Lambda_-(p)$ is open and dense in $p(T^*\mathbb{R}^n)$.

[1] The argument is an easy one-dimensional version of that in Section 5.

Since under our assumption $(p(T^*\mathbb{R}^n))^\circ \neq \emptyset$, an elementary version of the Morse-Sard theorem implies that $d\,\mathrm{Re}\,p$ and $d\,\mathrm{Im}\,p$ are independent on $p^{-1}(z)$ for z in a dense open set $\Omega \subset \Lambda(p) \setminus \Sigma_\infty(p)$. Lemma 3.1 then shows that $\Lambda_+(p) \cap \Omega = \Lambda_-(p) \cap \Omega$, completing the proof of the lemma. □

Remark. In the case of dimension 1 a different argument, based on elementary topological considerations, is needed, and some assumptions have to be made on p. To see that, consider, for instance,

$$p(x,\xi) = \frac{(\xi + ix)^2}{1 + x^2 + \xi^2}\,, \quad \{\mathrm{Re}\,p, \mathrm{Im}\,p\}(x,\xi) > 0,\ (x,\xi) \neq (0,0)\,.$$

For p's arising from Schrödinger operators considered in Section 1, we always have

$$\sum_{m \in p^{-1}(z)} \mathrm{sgn}\{\mathrm{Re}\,p, \mathrm{Im}\,p\}(m) = 0 \tag{3.2}$$

for a dense set of values z. In fact, $p(x,\xi) = p(x,-\xi) = z$, and the set of values z corresponding to $\xi \neq 0$ is dense in the set of values for which the bracket is nonzero. Now we simply notice that

$$\{\mathrm{Re}\,p, \mathrm{Im}\,p\}(x,\xi) = -\{\mathrm{Re}\,p, \mathrm{Im}\,p\}(x,-\xi)\,.$$

LEMMA 3.2′ *Suppose that $n = 1$ and, in addition to the general assumptions, each component of $\mathbb{C} \setminus \Sigma_\infty(p)$ has a nonempty intersection with $\complement\Lambda(p)$. Then the conclusions of Lemma* 3.2 *and* (3.2) (*for a dense set of values*) *hold.*

PROOF: Let Ω be a component of $\mathbb{C} \setminus \Sigma_\infty(p)$. Then

$$\iota \stackrel{\mathrm{def}}{=} \mathrm{var}\,\mathrm{arg}_{\gamma(z)}(p - z)$$

is independent of $z \in \Omega$ if $\gamma(z)$ is the positively oriented circle $|(x,\xi)| = R(z)$ with $R(z)$ large enough. For $z \in \Omega \setminus \Lambda(p)$, ι is zero (for mapping-degree reasons) and hence it is zero for all $z \in \Omega$. If $z \in \Lambda(p) \cap \Omega$ is a regular value, we get

$$0 = \iota = 2\pi \sum_{m \in p^{-1}(z)} \mathrm{sgn}\{\mathrm{Re}\,p, \mathrm{Im}\,p\}(m)\,,$$

so z belongs to both $\Lambda_+(p)$ and $\Lambda_-(p)$. □

In the remainder of the proof, there is no restriction on the dimension.

PROOF OF THEOREM 1.2′: We can assume that $z = 0$, and we follow the now-standard procedure of the *complex WKB construction* associated to a positive Lagrangian submanifold of the complexification of $T^*\mathbb{R}^n$. We start with the geometric construction of that submanifold. Since $\{\mathrm{Re}\,p, \mathrm{Im}\,p\} \neq 0$, we have $d_\xi p \neq 0$, and we can assume that $\partial_{\xi_1} p(x^0, \xi^0) \neq 0$. Let ϕ_0 be a real analytic function defined in a neighborhood of $y^0 \in \mathbb{R}^{n-1}$, $x^0 = (x_1^0, y^0)$, with the properties

$$\phi_0(y^0) = 0\,, \quad d\phi_0(y^0) = \eta^0\,, \quad \xi^0 = \left(\xi_1^0, \eta^0\right), \quad \mathrm{Im}\,d^2\phi_0(y^0) \gg 0\,,$$

where the Hessian $d^2 \operatorname{Im} \phi_0$ is well-defined at y^0 as $d \operatorname{Im} \phi_0 = 0$. We will make further assumptions on ϕ_0 later.

We then define $\Lambda_0 \subset T^*\mathbb{C}^n$ locally near (x^0, ξ^0) as follows:

$$\Lambda_0 = \left\{ \left(x_1^0, y; \xi_1(y), d_y\phi_0(y)\right) : p\left(x_1^0, y; \xi_1(y), d_y\phi_0(y)\right) = 0,\ \xi_1^0(y^0) = \xi_1^0 \right\},$$

where we know that the function $\xi_1(y)$ is locally defined and analytic from our condition $\partial_{\xi_1} p \neq 0$. Using holomorphic continuation, we obtain a locally defined submanifold of $T^*\mathbb{C}^n$, $\Lambda_0 \cap T^*\mathbb{R}^n = \{(x^0, \xi^0)\}$. This submanifold is isotropic with respect to the complex symplectic form, and its tangent spaces[2] are positive in the sense that (2.10) is satisfied without the condition on the dimension.

For $t \in \mathbb{C}$, $|t| < \epsilon$, the complex flow Φ_t exists by the Cauchy-Kovalevskaya theorem:

$$\Phi_t(z, \zeta) = (z(t), \zeta(t)), \quad z(0) = z,\ \zeta(0) = \zeta,$$
$$z'(t) = \partial_\zeta p(z(t), \zeta(t)), \quad \zeta'(t) = -\partial_z p(z(t), \zeta(t)),$$

that is, $\frac{d}{dt}(z(t), \zeta(t)) = H_p(z(t), \zeta(t))$, $\omega_{\mathbb{C}}(\bullet, H_p) = dp$.

We then define

$$\Lambda = \bigcup_{t \in \mathbb{C},\ |t| < \epsilon} \Phi_t(\Lambda_0) \subset T^*\mathbb{C}^n,$$

which is Lagrangian with respect to $\omega_{\mathbb{C}}$.

We now want to guarantee that tangent spaces to Λ are positive in the sense of (2.10). We first note that

$$i\omega_{\mathbb{C}}(\overline{tH_p}, tH_p) = i|t|^2\{\bar{p}, p\} = -2\{\operatorname{Re} p, \operatorname{Im} p\}|t|^2 > \gamma|t|^2 > 0$$

by the assumptions of the theorem. We have

$$T_{(x^0,\xi^0)}\Lambda = T_{(x^0,\xi^0)}\Lambda_0 + \operatorname{span}_{\mathbb{C}} H_p(x^0, \xi^0),$$
$$i\omega_{\mathbb{C}}(\overline{X + tH_p}, X + tH_p) = i\omega_{\mathbb{C}}(\overline{X}, X) + 2\operatorname{Im}(t\,dp^*(X)) + |t|^2 i\omega_{\mathbb{C}}(\overline{H_p}, H_p),$$

where $p^*(\rho) = \overline{p(\bar{\rho})}$. Hence, for positivity, we need to show that we can choose ϕ_0 so that for $X \in T_{(x^0,\xi^0)}\Lambda_0$,

$$|dp^*(X)|^2 < \alpha i\omega_{\mathbb{C}}(\overline{X}, X), \quad \alpha = -2\{\operatorname{Re} p, \operatorname{Im} p\}(x^0, \xi^0).$$

A calculation in local coordinates $(z_1, z'; \zeta_1, \zeta')$ shows that this follows from

$$\|A + \Phi B\|^2 < \alpha \min \operatorname{Spec}(\operatorname{Im} \Phi),$$
$$\Phi = \phi_0'', \quad A = |p'_{\zeta_1}|^{-1} i\left(p'_{\zeta_1}\overline{p}'_{z'} - \overline{p}'_{\zeta_1} p'_{z'}\right), \quad B = |p'_{\zeta_1}|^{-1} i\left(p'_{\zeta_1}\overline{p}'_{\zeta'} - \overline{p}'_{\zeta_1} p'_{\zeta'}\right).$$

The vectors A and B are real, and hence if $B \neq 0$ we can choose the complex matrix Φ so that $A + (\operatorname{Re}\Phi)B = 0$. This leaves us with

$$\|(\operatorname{Im}\Phi)B\|^2 < \alpha \min \operatorname{Spec}(\operatorname{Im}\Phi),$$

[2] These tangent spaces $T_\rho\Lambda_0$ are complex linear subspaces of $T_\rho\mathbb{C}^n$.

which can be arranged by making $\operatorname{Im}\Phi$ sufficiently small.[3]

From (2.11) we see that $\Lambda \subset \{p = 0\}$ is locally a graph, and since it is Lagrangian, a graph of a differential of a phase function ϕ. Since the tangent plane is positive, (2.11) shows that the Hessian of that phase function has a positive definite imaginary part:

$$\Lambda = \{(z, d_z\phi(z))\}\,, \quad p(z, d_z\phi) = 0\,,$$
$$\phi(x^0) = 0\,,\ \ d_z\phi(x^0) = \xi^0,\ \ \operatorname{Im} d_z^2\phi(x^0) \gg 0\,.$$

We also note that $\Lambda \cap T^*\mathbb{R}^n = \{(x^0, \xi^0)\}$, which corresponds to the fact that $\operatorname{Im} d_z\phi \neq 0$ for $z \neq x^0$.

Once the phase function has been constructed, we apply the usual WKB construction:

$$v(z, h) \sim e^{\frac{i\phi(z)}{h}} \sum_{j=0}^{\infty} a_j(z) h^j\,,$$

where we will want the coefficients a_j to be holomorphic near $z = x^0$ and to satisfy bounds $|a_j(z)| \leq C^j j^j$. They are constructed so that

$$\left(\sum_{j<1/(Ch)} p_j^w(z, hD_z) h^j \right) \left(e^{\frac{i\phi(z)}{h}} \sum_{j<1/(Ch)} a_j(z) h^j \right) = \mathcal{O}\big(e^{-\frac{1}{Ch}}\big)\,.$$

Here p^w denotes the Weyl quantization of a holomorphic symbol $p(z, \zeta)$ acting on holomorphic functions (compare to (2.1)):

$$p^w(z, hD_z)u = \frac{1}{(2\pi h)^n} \iint_{\Gamma_z} p\left(\frac{z+w}{2}, \zeta\right) e^{\frac{i}{h}\langle z-w, \zeta\rangle} u(w)\, dw\, d\zeta\,,$$

where the contour Γ_z is suitably chosen; see [24, sec. 4] for a discussion of the general case.

The transport equations for the a_j's then are

$$\sum_{k=1}^{n} \partial_{\zeta_k} p_0(z, d_z\phi(z)) \partial_{z_k} a_j(z) + i p_1(z, d_z\phi(z)) a_j = A_j(z)\,,$$

where $A_j(z)$ depends on the a_l's with $l < j$, and we put $a_0(x_1^0, y) = 1$. It is now classical that the solutions satisfy $|a_j| \leq C^j j^j$ near x^0; see [24, theorem 9.3]. The real quasi mode is obtained by restricting to the real axis and by truncating $v(z, h)$:

$$u(x, h) = \chi(x) v(x, h)\,, \quad \chi(x) = 1\,, \quad |x - x^0| < \delta\,, \quad \operatorname{supp}\chi \subset \mathbb{B}(x^0, 2\delta)\,,$$

where δ is small. Since the construction has shown that $\operatorname{Im}\phi \geq |x - x^0|^2/C$, the cutoff function χ does not destroy the exponential smallness of the error. □

[3] An alternative, and slicker, way of proceeding is by first observing that the positivity is invariant under affine linear canonical transformations. Using that fact and a multiplication by a nonvanishing factor, we can assume that $p = \xi_n - i x_n + \mathcal{O}((x, \xi)^2)$ and that $(x^0, \xi^0) = (0, 0)$. It is then straightforward to find $\phi(x)$ with the desired properties.

For completeness and later use in Section 6, we include a result on the discreteness of the spectrum.

PROPOSITION 3.3 *Suppose that* $p \in \mathcal{C}_{\mathrm{b}}^{\infty}(T^*\mathbb{R}^n)$. *Let* Ω *be an open, connected* (h-*independent*) *set satisfying*

$$\overline{\Omega} \cap \Sigma_\infty(p) = \emptyset\,, \quad \Omega \cap \complement\Sigma(p) \neq \emptyset\,.$$

Then $(p^w(x, hD) - z)^{-1}$, $0 < h < h_0(\Omega)$, $z \in \Omega$, *is a meromorphic family of operators with poles of finite rank.*

In particular, for h sufficiently small, the spectrum of $p^w(x, hD)$ *is discrete in any such set.*

PROOF: If Ω satisfies the assumptions of the proposition, then there exists $C > 0$ such that for every $z \in \Omega$, we have $|p(x, \xi) - z| > \frac{1}{C}$ if $|(x, \xi)| > C$. The assumption that $\Omega \cap \complement\Sigma(p) \neq \emptyset$ implies that for some $z_0 \in \Omega$, $(p(x, \xi) - z_0)^{-1} \in \mathcal{C}_{\mathrm{b}}^{\infty}(T^*\mathbb{R}^n)$. Let $\chi \in \mathcal{C}_{\mathrm{c}}^{\infty}(T^*\mathbb{R}^n; [0, 1])$ be equal to 1 in a sufficiently large bounded domain. The remarks above show that

$$r(x, \xi; z) = \chi(x, \xi)(z_0 - p(x, \xi))^{-1} + (1 - \chi(x, \xi))(z - p(x, \xi)^{-1}$$

is in $\mathcal{C}_{\mathrm{b}}^{\infty}(T^*\mathbb{R}^n)$. The symbol calculus reviewed in Section 2 then gives

$$\begin{aligned} r^w(x, hD, z)(z - p^w(x, hD)) &= I + \mathcal{O}_{L^2 \to L^2}(h) + K_1(z)\,, \\ (z - p^w(x, hD))r^w(x, hD, z) &= I + \mathcal{O}_{L^2 \to L^2}(h) + K_2(z)\,, \end{aligned}$$

where $K_j(z)$, $j = 1, 2$, are compact operators on $L^2(\mathbb{R}^n)$ depending holomorphically on z and vanishing for $z = z_0$.

By the analytic Fredholm theory, we conclude that $(z - p^w(x, hD))^{-1}$ is meromorphic in Ω for h sufficiently small. □

Remark. The same result holds for $P(h)$ of the form considered in Section 2 with $p_j \in \mathcal{C}_{\mathrm{b}}^{\infty}(T^*\mathbb{R}^n)$: the lower-order terms do not affect the meromorphy when h is small. We also comment on the case presented in Section 1.

Suppose that $m(x, \xi)$ is an admissible weight function, that is, a positive function on $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ satisfying

$$\forall X, Y \in \mathbb{R}^{2n} \quad 1 \leq m(X) \leq C\langle X - Y\rangle^N m(Y) \quad \text{for some fixed } C \text{ and } N.$$

Following [7] for symbols satisfying $|\partial_X^\alpha p(X)| \leq C_\alpha m(X)$, we can define operator $P = p^w(x, hD)$. In the analytic case we require that

$$|p(X)| \leq m(\mathrm{Re}\, X)\,, \quad |\mathrm{Im}\, X| \leq \frac{1}{C}\,.$$

In the example given in Section 1, we can take $m(x, \xi) = \langle\xi\rangle^2 + \langle x\rangle^p$.

Under an ellipticity assumption

$$|p(X)| \geq \frac{m(\mathrm{Re}\, X)}{C}\,, \quad |X| \geq C\,, \quad |\mathrm{Im}\, X| \leq \frac{1}{C}\,,$$

we obtain an invertibility: if $z_1 \notin \overline{p(\mathbb{R}^{2n})}$, then $P - z_1$ is invertible. If we define the operator

$$Q = (P - z_1)^{-1}(P - z_3)\,, \quad z_3 \neq z_1\,,$$

then the resolvents of Q and P are related by

$$(Q - \zeta)^{-1} = (1 - \zeta)^{-1}(P - z_1)\left(P - \frac{\zeta z_1 - z_3}{\zeta - 1}\right)^{-1}$$

so that the reduction of Schrödinger operators to the case of operators with bounded symbols was justified.

4 Decrease of Pseudospectrum by a Change of Norms

In this section we will prove Theorem 1.3 by using a dynamically defined function G that grows on the bicharacteristics of $\operatorname{Re} qp$ for some $0 \neq q \in C^\infty$.

In the analytic case we will now use the microlocally weighted spaces $H(\Lambda_{tG})$, the construction of which was recalled in Section 2, to decrease the pseudospectrum by changing the norm on $L^2(\mathbb{R}^n)$ in an h-dependent way. In particular, this will show that under assumptions (1.6) and (1.7), the spectrum is separated from the boundary of the pseudospectrum.

4.1 Construction of Microlocal Weights

We start with the following lemma:

LEMMA 4.1 *Assume that* (1.6) *and* (1.7) *hold for* $z_0 \in \partial\Sigma(p)$. *Then there exists* $q(x,\xi) \in C_{\mathrm{b}}^\infty$ *such that* $q \neq 0$ *on* $p^{-1}(z_0)$ *and*

$$\operatorname{Im}(q(p - z_0)) \geq 0 \text{ near } p^{-1}(z_0)\,, \quad |d\operatorname{Re}(q(p - z_0))| \geq c > 0 \text{ on } p^{-1}(z_0)\,. \tag{4.1}$$

PROOF: By subtracting z_0 from p, we may assume $z_0 = 0$. Now, since $w_0 \in \partial\Sigma(p)$, we find that $d \operatorname{Re} p(w_0)$ and $d \operatorname{Im} p(w_0)$ are linearly dependent. Take $w_0 \in p^{-1}(0)$ and the semi-bicharacteristic $\gamma_0 \subset p^{-1}(0)$ through w_0, that is, the flow out of $H_{\operatorname{Re}(\lambda p)}$ for λ appearing in condition (1.7); we find that this semi-bicharacteristic is compact. In a neighborhood of γ_0, we have a defining function $\eta(x,\xi)$ of $(\operatorname{Re}\lambda p)^{-1}(0)$ such that η is real, $\eta = 0$, and $d\eta \neq 0$ when $\operatorname{Re}(\lambda p) = 0$. This function is well-defined up to nonvanishing real factors. By completing $\xi_1 = \eta$ to a symplectic coordinate system and using the Malgrange preparation theorem, with a partition of unity, as in the proof of [14, theorem 26.4.13], we may write $\xi_1 = qp + r(x,\xi')$ in a neighborhood of γ_0. By completing x_1, $\xi_1 - \operatorname{Re} r(x,\xi')$ to a symplectic coordinate system near γ_0, we obtain that $r = i\varrho$ with real-valued $\varrho(x,\xi')$. In fact, we have that $\partial_{\xi_1}\varrho \equiv \{x_1, \varrho\} \equiv 0$ is preserved. Thus, we obtain the normal form $qp = \widetilde{p} = \xi_1 - i\varrho$ such that $\operatorname{Im}(qp) = -\varrho$ in a neighborhood ω_0 of γ_0.

From invariance of the Poisson bracket, we find that $\partial_{x_1}\varrho = -|q|^2\{\operatorname{Re} p, \operatorname{Im} p\}$ cannot vanish identically on γ_0 by (1.7); thus $\varrho \neq 0$ at some parts of γ_0. Since $0 \in \partial\Sigma(p)$, we find that $\pm\varrho \geq 0$ for a choice of sign. In fact, if ϱ changes sign, then we can find a piecewise linear curve $\Gamma \subset \omega_0$ such that the winding number of $\widetilde{p}(\Gamma)$ is nonzero. Since ω_0 is simply connected, we find that $p(\Gamma) = q\widetilde{p}(\Gamma)$ has nonzero winding number, which gives $0 \in \Sigma(p)^\circ$ and a contradiction.

If the sign of $\operatorname{Im}(qp) = -\varrho \le 0$, we change ξ_1, q, and ϱ to $-\xi_1$, $-q$, and $-\varrho$ to obtain $\operatorname{Im}(qp) \ge 0$. Then we find that q is well-defined up to positive factors on $p^{-1}(0)$. In fact, if we take two overlapping neighborhoods on $p^{-1}(0)$ with normal forms having nonnegative imaginary parts, then the quotient of the different q must be positive on $p^{-1}(0)$. By taking a partition of unity, we obtain the result, since $d\operatorname{Im}(qp) = 0$ on $p^{-1}(0)$. $\square$

Observe that Lemma 4.1 or, more specifically, $H_{\operatorname{Re}(qp)}$ gives an orientation of the bicharacteristics of $p^{-1}(z_0)$. This orientation allows us to construct a global weight G by the next lemma.

LEMMA 4.2 *Suppose that* (4.1) *holds with* $q(x,\xi) \in \mathcal{C}_\mathrm{b}^\infty$. *Then there exists* $G \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^{2n};\mathbb{R})$ *such that* $H_{\operatorname{Re}(qp)}G(\rho) > 0$ *for every* $\rho \in p^{-1}(0)$.

PROOF: Assumption (1.7) gives a seemingly stronger statement:

$$\exists T_0 > 0 \quad \forall (x,\xi) \in (\operatorname{Re}(qp))^{-1}(0) \quad \exists 0 < t < T_0 \quad \text{such that } \operatorname{Im}(qp)(\exp(tH_{\operatorname{Re}(qp)})(x,\xi)) \neq 0 \tag{4.2}$$

where q is given by Lemma 4.1. This last condition seems different from (1.7), since we are putting $\lambda = q$. However, $d(\operatorname{Im}(qp)) = 0$ on $p^{-1}(0)$ by Lemma 4.1; thus (1.7) for some λ implies (4.2). It now allows us to construct a global weight G. Indeed, we first construct G locally: Let $\gamma(t)$, $0 \le t \le T_1$ ($0 \le T_1 < T_0$), be a maximal $H_{\operatorname{Re}(qp)}$ orbit in $p^{-1}(0)$. Then we can find a real-valued $G \in \mathcal{C}_\mathrm{c}^\infty$ with support in a small neighborhood of the image of γ such that $H_{\operatorname{Re}(qp)}G \ge 0$ on $p^{-1}(0)$ with strict inequality on the image of γ. We then get the G of the lemma by taking a finite sum of such local G's. $\square$

4.2 The $\mathcal{C}^\infty$ Case

Here we will assume, without loss of generality, that $z_0 = 0$. If we consider points in $p^{-1}(0)$, we may replace p by qp. Let

$$C_1 h \le \epsilon \le C_2 h \log\frac{1}{h} \tag{4.3}$$

where $C_1 > 0$ is large enough. We shall derive an estimate for the

$$P_\epsilon(h)u = v$$

when $\mathrm{WF}_h(u)$ is contained in a small neighborhood of $p^{-1}(0)$, and where

$$P_\epsilon(h) \stackrel{\text{def}}{=} e^{\frac{\epsilon G}{h}} P(h) e^{-\frac{\epsilon G}{h}} = e^{\frac{\epsilon}{h}\operatorname{ad}_G} P(h) \sim \sum_0^\infty \frac{\epsilon^k}{k!}\left(\frac{1}{h}\operatorname{ad}_G\right)^k (P(h)),$$
$$G = G^w(x,hD).$$

We note that the assumption on ϵ and the boundedness of ad_G /h show that the expansion makes sense. We may replace $P(h)$ by $q^w P(h)$ where q is given by Lemma 4.1, but first for simplicity we consider the case $q \equiv 1$; thus $\mathrm{Im}\, p \geq 0$ near $p^{-1}(0)$. The operators $\exp(\epsilon G/h)$ are pseudodifferential in an exotic class $S_\delta^{C_2}$ for any $\delta > 0$ (see [7]), but that is not relevant here.

Our method is inspired by the work of Unterberger [31], who used the theory of pseudodifferential operators of variable order (Unterberger and Bokobza [32]) to study propagation of regularity for partial differential equations. Following [31], we point out that Malgrange [18] had earlier used spaces of functions of variable regularity in the case of equations with constant coefficients. In a related context of the absence of resonances, Martinez [21] recently used similar methods.

Dropping the h in $P(h)$ and using the same letters for operators and the corresponding symbols, we see that

$$P_\epsilon = P + i\epsilon\{p, G\} + \mathcal{O}(\epsilon^2) = p + i\epsilon\{p, G\} + \mathcal{O}(h + \epsilon^2)$$

so that

$$\mathrm{Re}\, P_\epsilon = \mathrm{Re}\, p - \epsilon\{\mathrm{Im}\, p, G\} + \mathcal{O}(h + \epsilon^2)\,,$$
$$\mathrm{Im}\, P_\epsilon = \mathrm{Im}\, p + \epsilon\{\mathrm{Re}\, p, G\} + \mathcal{O}(h + \epsilon^2)\,.$$

Since we put $q = 1$, Lemma 4.1 gives

$$\mathrm{Im}\, p \geq 0 \tag{4.4}$$

near $p^{-1}(0)$. Hence

$$\mathrm{Im}\, P_\epsilon \geq \frac{\epsilon}{C} + \mathcal{O}(h + \epsilon^2)\,. \tag{4.5}$$

We then consider

$$J = \left(\frac{1}{2i}(P_\epsilon - P_\epsilon^*)u, u\right) = \mathrm{Im}(P_\epsilon u, u)$$

so that

$$|J| \leq C\|P_\epsilon u\|\|u\|\,. \tag{4.6}$$

On the other hand, the symbol of

$$\frac{1}{2i}\big(P_\epsilon - P_\epsilon^*\big)$$

is equal to the left-hand side of (4.5) plus $\mathcal{O}(h)$, so (4.5) and the sharp Gårding inequality (see [7, theorem 7.12] or apply (2.8) with $G = 0$ and $p_1 = \mathrm{Im}\, P_\epsilon$, $p_2 = 1$) imply

$$J \geq \frac{\epsilon}{2C}\|u\|^2\,,$$

where we recall that u had its wave front set close to $p^{-1}(0)$ where all our estimates are valid. Combining this with (4.6), we get

$$\|u\| \leq \frac{2C_2}{\epsilon}\|P_\epsilon u\|\,. \tag{4.7}$$

When $WF_h(u)$ is away from $p^{-1}(0)$, then ellipticity (in the semiclassical sense) gives $\|u\| \leq C\|P_\epsilon u\| + \mathcal{O}(h^\infty)\|u\|$, which shows that (4.7) holds for any $u \in \mathcal{C}_c^\infty$. When $\epsilon \sim Mh$ and $M \gg 1$, we see that $\|P^{-1}\| \sim \|P_\epsilon^{-1}\| = \mathcal{O}(\frac{1}{h})$, as stated in the remark after Theorem 1.3.

For larger ϵ we cannot compare the norms in an h-independent way but $\sigma(P) = \sigma(P_\epsilon)$. In view of (4.3), where $C_2 > 0$ can be arbitrarily large, we conclude that for every $C > 0$, we have

$$D\left(0, Ch\log\frac{1}{h}\right) \cap \sigma(P) = \varnothing\,, \quad 0 < h < h(C)\,,$$

when $h(C) > 0$ is sufficiently small.

In the case $q \not\equiv 1$, we replace $P(h)$ with $q^w(x, hD_x)P(h)$ having principal part $(qp)^w(x, hD_x)$. Then $(qP)_\epsilon = q_\epsilon P_\epsilon$, and since q_ϵ is an elliptic factor, we get the same results.

Remark. In the assumptions of Theorem 1.3, we took $z_0 \in \partial\Sigma(p)$. The other assumptions show that if we replace $\Sigma(p)$ by $\Lambda(p)$ in our assumptions, the seemingly stronger assumption still holds. In fact, suppose that there exists a sequence $z_j \to 0$ such that $z_j \in \Sigma(p) \setminus (\Lambda(p) \cup \Sigma_\infty(p))$. That means that $\{p, \bar{p}\} \restriction_{p^{-1}(z_j)} = 0$ and that trajectories of $H_{\mathrm{Re}\, p}$ stay in $p^{-1}(z_j)$ for all times and are contained in a bounded set. Taking the limit of the points of these trajectories as $z_j \to 0$ gives a contradiction to (1.7).

Example. We want to indicate why the result involving $h\log(\frac{1}{h})$ is optimal. As mentioned before, the proof of our argument is similar to arguments showing the absence of resonances [20]. In that setting it is well-known that regularity of the potential determines the size of the pole-free region. It is particularly easy to see in the case of dimension 1 [33]. We can adapt methods of one-dimensional scattering in the following simple example:

$$P(h) = (hD_x)^2 - iW(x)\,, \quad x \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}\,,$$
$$W(x) = W(-x) \geq 0\,, \quad \operatorname{supp} W \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

For a potential V, $V(-x) = V(x)$, let $A(\lambda, h, V)^{-1}$ be the transmission coefficient for $(hD_x)^2 + V(x) - \lambda^2$. In the notation of [33], it is given by $\widehat{X}(\xi)/i\xi$, $\xi = \lambda/h$, with the potential $p(x) = h^{-2}V(x)$. Expressing the monodromy operator

$$M(\lambda, h, V) : (u(-\pi), hD_x u(-\pi)) \longmapsto (u(\pi), hD_x u(\pi))$$

in terms of the scattering matrix, we obtain

$$\begin{aligned} \lambda^2 \in \sigma(P(h)) &\Leftrightarrow 1 \in \sigma(M(\lambda, h, -iW)) \\ &\Leftrightarrow A(\lambda, h, -iW)e^{\frac{2\pi i\lambda}{h}} + A(-\lambda, h, -iW)e^{-\frac{2\pi i\lambda}{h}} = 2\,. \end{aligned}$$

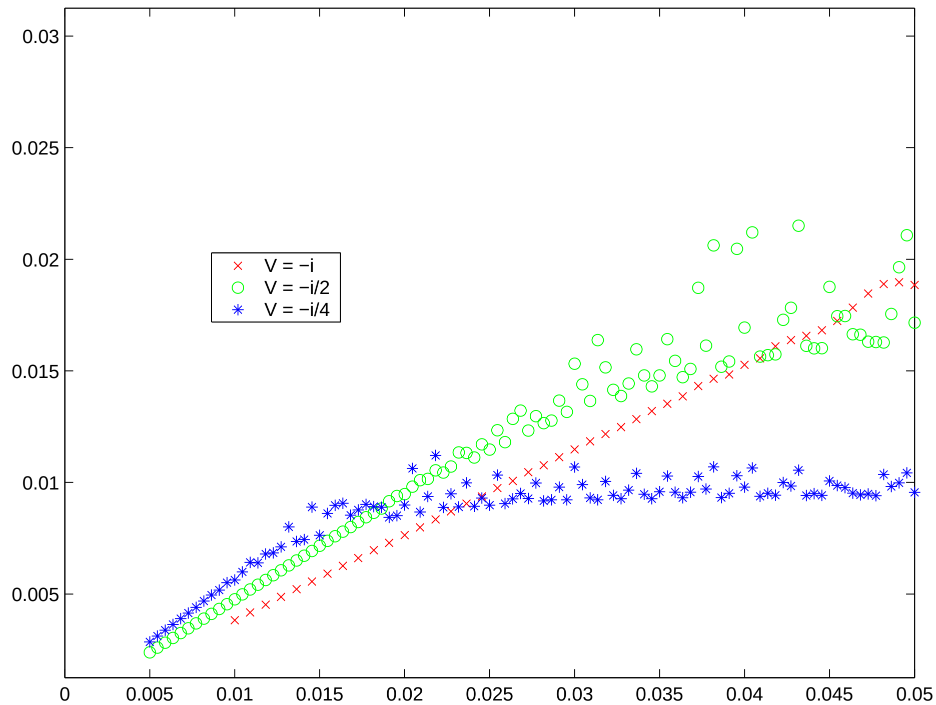

FIGURE 4.1. Plot of imaginary parts of eigenvalues of $(hD_x)^2 + V(x)$, $V(x) = -iW(x)$, where $W(x)$ is a step potential, as functions of h. The real part is close to 2, and the figure shows numerical results based on (4.8).

In the simplest case of $V(x) = V_0 \mathbb{1}_{[-a,a]}$, we have the elementary formula

$$A(\lambda, h, V) = e^{\frac{ia}{h}\lambda} \left(\cos \frac{a\rho}{h} - i \frac{\lambda^2 + \rho^2}{2\lambda\rho} \sin \frac{a\rho}{h} \right), \quad \rho = \left(1 - \frac{V_0}{\lambda^2}\right)^{\frac{1}{2}},$$

which leads to a transcendental equation for λ,

$$\begin{gathered}(1+r)^2 \cos((\alpha - r)\xi) - (1-r)^2 \cos((\alpha + r)\xi) = 4r\,, \\ \xi = \frac{a\lambda}{h}, \quad r = \left(1 - \frac{V_0}{\lambda^2}\right)^{\frac{1}{2}}, \quad \alpha = 1 + \frac{2\pi}{a}\,. \end{gathered} \tag{4.8}$$

Figure 4.1 shows imaginary parts of numerical solutions $\lambda \sim 2$ to (4.8) for three different values of V_0. As $h \to 0$ we see linear dependence. There is no $\log(\frac{1}{h})$ improvement due to the lack of continuity. We expect that for more regular potentials, the imaginary parts will behave as $kh \log(\frac{1}{h})$ with k related to the number of continuous derivatives.

4.3 The Analytic Case

To apply the theory of weighted spaces $H(\Lambda_{tG})$ reviewed in Section 2, we need one more lemma.

LEMMA 4.3 *Suppose that* (1.6) *and* (1.7) *hold, and that* G *is given by Lemma* 4.2. *In the notation of* (2.7) *we have, for sufficiently small* $t > 0$,

$$|p\restriction_{\Lambda_{tG}}| > \frac{t}{C}\,. \tag{4.9}$$

PROOF: Assume that $\rho_0 \in p^{-1}(0)$ and $\lambda_0 = q(\rho_0)$ where q is given by Lemma 4.1; then we find that $d\,\mathrm{Im}(\lambda_0 p)(\rho_0) = 0$. Thus, $d\,\mathrm{Re}(\lambda_0 p)(\rho_0) \neq 0$, so for every $\epsilon > 0$, there is a neighborhood W of ρ_0, such that if $\rho \in W$, then

$$\mathrm{Im}(\lambda_0 p)(\rho) \leq \epsilon|\mathrm{Re}(\lambda_0 p)(\rho)|\,. \tag{4.10}$$

Now,

$$\begin{aligned}\lambda_0 p(\rho + itH_G) &= \lambda_0 p(\rho) - itH_{\lambda_0 p}G(\rho) + \mathcal{O}(t^2)\\ &= \lambda_0 p(\rho) - itH_{\mathrm{Re}(\lambda_0 p)}G + tH_{\mathrm{Im}(\lambda_0 p)}G + \mathcal{O}(t^2)\,.\end{aligned}$$

It follows that

$$\mathrm{Im}(\lambda_0 p)(\rho + itH_G(\rho)) = \mathrm{Im}(\lambda_0 p)(\rho) - tH_{\mathrm{Re}(\lambda_0 p)}G(\rho) + \mathcal{O}(t^2)\,.$$

Since $H_{\mathrm{Re}(\lambda_0 p)}G(\rho_0) > 0$, we find, for $|\rho - \rho_0|$ small enough,

$$\begin{aligned}&\big(\mathrm{Im}(\lambda_0 p) - \epsilon|\mathrm{Re}(\lambda_0 p)|\big)(\rho + itH_G(\rho)) \leq \\ &\qquad -\frac{1}{C}t + \mathcal{O}(t|\rho - \rho_0|) + \mathcal{O}(\epsilon t) \leq -\frac{1}{2C}t\,,\end{aligned} \tag{4.11}$$

where we also used (4.10). Hence,

$$|\lambda_0|^2\,|p\restriction_{\Lambda_{tG}}|^2 = |\lambda_0 p\restriction_{\Lambda_{tG}}|^2 \geq \left(\frac{t}{C_1} - \epsilon|\mathrm{Re}(\lambda_0 p)|\right)^2 + (\mathrm{Re}(\lambda_0 p))^2 \geq \frac{t^2}{C_1^2}\,,$$

which completes the proof. □

As a consequence of the last lemma, we see that 0 is a removable point of the pseudospectrum. To see this, we recall that $H(\Lambda_{tG})$ is equal to L^2 as a space and that the norms are equivalent for every fixed h (but not uniformly with respect to h). The spectrum of $P(h)$ therefore does not depend on whether we realize this operator on L^2 or on $H(\Lambda_{tG})$. We conclude that 0 has an h-independent neighborhood that is disjoint from the spectrum of $P(h)$ when h is small enough. All this is summarized in the following result, which is an immediate consequence of Lemma 4.3 and (2.9):

THEOREM 1.3′ *Suppose that* $z_0 \in \partial\Lambda(p) \setminus \Sigma_\infty(p)$ *and that* (1.6) *and* (1.7) *hold. In the notation of Lemma* 4.2, *let us introduce the IR manifold*

$$\Lambda_{tG} = \left\{\rho + itH_G(\rho);\, \rho \in \mathbb{R}^{2n}\right\} \quad \textit{for } t > 0 \textit{ small enough.}$$

If

$$P(h) \sim \sum_j h^j p_j^w(x, hD), \quad p_0 = p,$$

where the p_j *satisfy the assumptions of Section* 2*, then*

$$P(h) - z_0 : H(\Lambda_{tG}) \longrightarrow H(\Lambda_{tG})$$

has a bounded inverse for h *small enough. In particular, for* δ *small enough but independent of* h*,*

$$\sigma(P(h)) \cap D(z_0, \delta) = \varnothing, \quad 0 < h < h_0.$$

5 Proof of Theorem 1.4

We will see (Lemma 5.1) that the theorem holds under the seemingly weaker assumption that $z_0 \notin \Lambda(p)^\circ$. We first observe that the property of being of finite type and the order of the symbol are invariant under multiplication with elliptic factors (as is condition (P); see [14, theorem 26.4.12]). In fact, if $0 \neq q \in \mathcal{C}^\infty$, then the repeated Poisson brackets of $\mathrm{Re}\, qp$ and $\mathrm{Im}\, qp$ of order $\leq j$ is a linear combination with smooth coefficients of those of p_1 and p_2 and vice versa; see [14, sec. 27.2].

Since we rely on a localization argument based on Weyl calculus of pseudodifferential operators, we will follow [14, sec. 18.5] and introduce

$$g_h(dx, d\xi) = |dx|^2 + h^2 |d\xi|^2. \tag{5.1}$$

We then find that $p(x, h\xi) \in S(1, g_h)$.

5.1 Reduction to Normal Form

Assume that $z_0 \in \Sigma(p)$ satisfies (1.6) by subtracting z_0; we may assume that $z_0 = 0$. Let $w_0 = (x_0, \xi_0) \in p^{-1}(0)$; then, since $|dp(w_0)| \neq 0$, we may assume that $\partial_x p = 0$ and $\partial_{\xi_j} p = 0$, $j > 0$, at w_0 by a linear change of coordinates. By making a symplectic change of variables and using the Malgrange preparation theorem, we obtain

$$p = q(\xi_1 + i f(x, \xi')) = q\widetilde{p}, \quad \xi = (\xi_1, \xi'), \tag{5.2}$$

in a neighborhood Ω of w_0 (see [14, theorem 21.3.6]). Here $0 \neq q \in \mathcal{C}_\mathrm{b}^\infty$ and $f(x, \xi') \in \mathcal{C}_\mathrm{b}^\infty$ is real valued.

By the invariance we find that $\widetilde{p}_I(w_0) \neq 0$ for some I such that $|I| = k + 1$, where k is the order of p at w_0, which is less than or equal to the order of $z_0 = 0$.

LEMMA 5.1 *Assume that* p *is of the form* (5.2)*,* $z_0 \notin \Lambda(p)^\circ$ *is of finite type, and* $p(w_0) = z_0$*. Then we obtain that* $z_0 \in \partial\Lambda(p) \cap \partial\Sigma(p)$*,* $\pm f \geq 0$*,* k *is even, and*

$$(H_{\mathrm{Re}(\lambda p)})^k \, \mathrm{Im}(\lambda p)(w_0) \neq 0 \quad \textit{for almost all } \lambda \in \mathbb{C}. \tag{5.3}$$

Thus, in a neighborhood of w_0 we find that p satisfies condition (P).

PROOF: As usual, we subtract a constant so that $z_0 = 0$ and choose coordinates in which $w_0 = 0$ and $(x, \xi) = (x_1, z)$. We find from the definition of $k(w)$ that $H^j_{\mathrm{Re}\, p} \operatorname{Im} p(0) = \partial^j_{x_1} f(0) = 0$ when $j < k$.

We shall now show that $d_z \partial^j_{x_1} f(0) = 0$ when $j < \frac{k}{2}$. In fact, if $d_z \partial^j_{x_1} f(0) \neq 0$ for some $j < \frac{k}{2}$, then

$$|d_z f(x_1, 0; 0)| \geq c|x_1|^j, \quad 0 < |x_1| < c\,.$$

Since $\pm f \geq 0$ implies that

$$|d_z f| \leq \sqrt{2|f| \|\partial_z^2 f\|_\infty}$$

and $|f(x_1, 0; 0)| \leq C|x_1|^k$, we find in this case that $z \mapsto f(x_1, z)$ changes sign in a neighborhood ω_{x_1} of the origin for any $0 < |x_1| \ll 1$. Now $\{p, \overline{p}\} = \frac{1}{2i} \partial_{x_1} f$, and since $d_z \partial^j_{x_1} f(0) \neq 0$, we have that $|d_z \partial_{x_1} f(x_1, 0; 0)| \geq c|x_1|^{j-1}$. Thus, for small enough $|x_1| > 0$, we find that $H_{\mathrm{Re}\, p} \operatorname{Im} p = \partial_{x_1} f \neq 0$ almost everywhere in ω_{x_1}, which contradicts the assumption that $0 \notin \Lambda(p)^\circ$. Hence we have shown that

$$d_z \partial^j_{x_1} f(0) = 0\,, \quad j < \frac{k}{2}\,, \tag{5.4}$$

which implies that

$$\widetilde{p}_I(0) \neq 0 \Longrightarrow I = (1, \dots, 1, 2)\,; \tag{5.5}$$

that is, the only iterated bracket of order $k + 1$ that does not vanish at 0 is $\partial^k_{x_1} f$.

To see this, let us consider a general bracket $\widetilde{p}_I$ of order $k + 1$, which is a sum of terms of the form

$$H_{\partial^{j_1}_{x_1} f} \cdots H_{\partial^{j_\nu}_{x_1} f} \partial^{j_{\nu+1}}_{x_1} f\,, \quad \nu + j_1 + \cdots + j_{\nu+1} \leq k\,,$$

where we note that $H_{\partial^{j_\ell}_{x_1} f}$ is a vector field in z only, since f is independent of ξ_1. We want to show that $\nu = 0$. Otherwise, (5.4) gives $j_1 \geq \frac{k}{2}$, and consequently $j_2 + \cdots + j_{\nu+1} \leq \frac{k}{2} - \nu < \frac{k}{2}$. To get a nonvanishing contribution, in the composition of vector fields $H_{\partial^{j_\ell}_{x_1} f}$, all the derivatives have to fall on the coefficients of $H_{\partial^{j_{\ell'}}_{x_1} f}$, $\ell' > \ell$, since by (5.4) all these vector fields vanish at 0. Hence we will always have a factor containing z derivatives of $\partial^{j_{\nu+1}}_{x_1} f$, and as $j_{\nu+1} < \frac{k}{2}$, it vanishes at 0. Hence $\nu = 0$ and we have established (5.5).

If f changes sign in a neighborhood of 0, then we find as in the proof of Lemma 4.1 that $0 \notin \partial \Sigma(p)$. This also implies that almost every value of p near w_0 is in $\Lambda(p)$. We obtain that $\pm f \geq 0$, k is even, and $z_0 \in \partial \Lambda(p)$. An elementary computation shows that

$$(H_{\mathrm{Re}(\lambda p)})^k \operatorname{Im}(\lambda p)(0) = (\operatorname{Re} \lambda)^{k-1} |\lambda|^2 \partial^k_{x_1} f(0)\,,$$

which gives the result. ☐

After possibly switching x_1 to $-x_1$ and multiplying by -1, we may assume that $f \geq 0$, $k = 2l$ is even, and

$$\partial_{x_1}^{2l} f(0) > 0\,. \tag{5.6}$$

By choosing a suitable cutoff function $\psi(x', \xi') \in \mathcal{C}_c^\infty(T^*\mathbb{R}^{n-1})$ supported near $(0, \xi_0')$ such that $0 \leq \psi(x', \xi') \leq 1$ and replacing $f(x, \xi')$ by $\psi(x', \xi') f(x, \xi') + x_1^{2l}(1 - \psi(x', \xi'))$, we may assume that $f \in \mathcal{C}^\infty$ is uniformly bounded and

$$\partial_{x_1}^{2l} f(x, \xi') \geq c > 0 \quad \text{when } |x_1| \leq c_0\,. \tag{5.7}$$

By a change of variables we may assume that $c_0 = 1$, and by cutting off where $|x_1| > 1$, we may assume that $f \in \mathcal{C}_b^\infty(\mathbb{R} \times T^*\mathbb{R}^{n-1})$.

Since $0 \notin \Sigma_\infty(p)$, we may take $0 \leq \phi_j \in \mathcal{C}_c^\infty(T^*\mathbb{R}^n)$, $j = 1, \dots, N$, and $\phi_0 = 1 - \sum_{1 \leq j \leq N} \phi_j$ such that p is on the form (5.2) with f satisfying (5.7) in $\operatorname{supp} \phi_j$, $j > 0$, and $|p| \geq c_0 > 0$ on $\operatorname{supp} \phi_0$. Then we obtain that

$$\left\| \phi_0^w(x, hD_x)u \right\| \leq C(\| p^w(x, hD_x)u\| + h\|u\|)$$

by using that $p^{-1}(x, h\xi) \in S(1, g_h)$ on $\operatorname{supp} \phi_0$.

We find from Proposition 5.2 and (5.7) that

$$\left\| \phi_j^w(x, hD_x)u \right\| \leq Ch^{-\frac{k}{k+1}}\big(\| p^w(x, hD_x)u\| + h\|u\|\big)\,, \qquad j > 0\,,$$

for $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ supported where $|x_1| \leq 1$, since

$$\widetilde{p}^w(x, hD_x)\phi_j^w(x, hD_x) \cong \phi_j^w(x, hD_x)(q^{-1})^w(x, hD_x)\, p^w(x, hD_x) \tag{5.8}$$

modulo $\operatorname{Op} S(h, g_h)$. This gives

$$\|u\| \leq \sum_{j=0}^{N} \|\phi_j^w(x, hD_x)u\| \leq C_0 h^{-\frac{k}{k+1}}\big(\| p^w(x, hD_x)u\| + h\|u\|\big)\,;$$

thus for small enough $h > 0$, we find that

$$\|u\| \leq C_0 h^{-\frac{k}{k+1}} \| p^w(x, hD_x)u\| \tag{5.9}$$

for $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ supported where $|x_1| \leq 1$. Since this estimate can be perturbed with terms in $\operatorname{Op} S(h, g_h)$ for small enough h, we obtain Theorem 1.4.

The estimates for the localized operators will be proven in the next subsection.

5.2 The Model Operator

In this section we shall consider the subelliptic model operator

$$P(h) = hD_t + i f^w(t, x, hD_x)\,, \qquad 0 \leq f \in \mathcal{C}_b^\infty(\mathbb{R} \times T^*\mathbb{R}^{n-1}),\ 0 < h \leq 1\,, \tag{5.10}$$

which we assume satisfies (5.11).

PROPOSITION 5.2 *Assume $P(h)$ is given by* (5.10), *where $f \in \mathcal{C}_{\mathrm{b}}^{\infty}(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ satisfies*

$$\left|\partial_t^k f(t,x,\xi)\right| \ge c > 0\,, \quad |t| \le 1,\ \forall (x,\xi) \in T^*\mathbb{R}^n. \tag{5.11}$$

Then we obtain

$$\|u\| \le Ch^{-\frac{k}{k+1}} \|P(h)u\| \tag{5.12}$$

for $u \in \mathcal{C}_{\mathrm{c}}^{\infty}$ supported where $|t| \le 1$.

PROOF: By the nonnegativity of $f \in \mathcal{C}_{\mathrm{b}}^{\infty}$, we obtain from [13, lemma 7.7.2] that

$$|\partial_x f|^2 + |\partial_\xi f|^2 \le Cf\,, \quad |t| \le 1\,. \tag{5.13}$$

Let $f_h(t,x,\xi) = f(t,x,h\xi)$; then $P(h) = hD_t + if_h^w(t,x,D_x)$ where $f_h \in S(1,g_h)$ with g_h given by (5.1). We shall introduce a new symbol class adapted to f_h. By (5.13) we obtain that $|\partial_x f_h|^2 + h^{-2}|\partial_\xi f_h|^2 \le C|f_h|$, which means that

$$|f_h|_1^{g_h} \le C\sqrt{f_h}\,, \quad |t| \le 1\,. \tag{5.14}$$

Let

$$m(t,w) = f_h(t,w) + h^{\frac{k}{k+1}} \ge h^{\frac{k}{k+1}} \tag{5.15}$$

and $g_{m(t,w)} = g_h/m(t,w)$. Then it follows from (5.14) that g_m is uniformly σ-temperate, $g_m/g_m^\sigma = h^2 m^{-2} \le h^{2/(k+1)}$, and m is a weight for g_m uniformly when $0 < h \le 1$. In fact, we obtain from (5.14) that

$$|f_h(t,w) - f_h(t,w_0)| \le C\rho m(t,w_0) \quad \text{when } g_{m(t,w_0)}(w-w_0) \le \rho^2,$$

since $|f_h|_2^{g_h} \le C$ and $g_h(w-w_0) \le \rho^2 m(t,w_0)$, which gives the uniform slow variation. Since

$$g^\sigma_{m(t,w_1)}(w_1 - w_0) = h^{-2}m^2(t,w_1)g_{m(t,w_1)}(w_1-w_0) \ge ch^{-\frac{2}{k+1}}$$

when $g_{m(t,w_0)}(w_1 - w_0) \ge c$, and

$$\frac{g_{m(t,w_1)}}{g_{m(t,w_0)}} = \frac{m(t,w_0)}{m(t,w_1)} \le Ch^{-\frac{k}{k+1}},$$

we obtain that g_m is uniformly σ-temperate. It also follows from (5.14) that $f_h \in S(m,g_m)$, since $|f_h|_1^{g_m} = \sqrt{m}|f_h|_1^{g_h}$ and $|f_h|_j^{g_m} \le C_j m^{j/2}|f_h|_j^{g_h} \le C_j m$ when $j \ge 2$. By changing f where $|t| > 0$, we may assume that (5.11) is satisfied for all t. We shall prove a simple estimate for the one-dimensional operator.

LEMMA 5.3 *Let $P = hD_t + if(t)$, where $0 \le f(t) \in C^k(\mathbb{R})$ and $1 \le f^{(k)}(t)$ for all $t \in \mathbb{R}$. Then k is even and*

$$\left\|m^{\frac{1}{2}}u\right\| \le C_1\left\|m^{-\frac{1}{2}}Pu\right\|\,, \quad u(t) \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R})\,, \tag{5.16}$$

where $m(t) = f(t) + h^{k/(k+1)}$.

PROOF: By Taylor's formula we have

$$f(t) = \sum_{j=0}^{k-1} f^{(j)}(0)\frac{t^j}{j!} + t^k \int_0^1 f^{(k)}(\theta t)\frac{(1-\theta)^{k-1}}{(k-1)!}\,d\theta \geq \frac{t^k}{2}\,, \quad t \gg 1\,,$$

and if k is odd, we similarly find $f(t) \leq t^k/2$ for $t \ll -1$. Thus, k is even and $f(t) \geq t^k/2$ for $|t| \gg 1$. Now $t \to f(t)$ can only have at most k zeros t_i (counted with multiplicity) since if $f(t_j^0) = 0$, $t_1^0 \leq t_2^0 \leq \cdots$, for $j = 1, \dots, k+1$, then there exists a t_j^1 such that $t_j^0 \leq t_j^1 \leq t_{j+1}^1$, $j = 1, \dots, k$, such that $f'(t_j^1) = 0$. By iterating this argument, we obtain that $t \mapsto f^{(k)}(t)$ has a zero, which is a contradiction.

Next we show that there exists $C_k > 0$ depending only on k and the bounds on $f^{(k)}$ such that

$$f(t) \leq \delta \leq \frac{1}{C_k} \implies t \in \omega_\delta = \bigcup_i \left\{t : |t - t_i| < C_k \delta^{\frac{1}{k}}\right\}. \tag{5.17}$$

Thus, assume that $f(t) \leq \delta$ when $|t - t_j| \leq K\delta^{1/k}$ with $1 \ll K \ll \delta^{-1/k}$, where we may assume that $t_j = 0$. We introduce the rescaled function

$$f_\delta(t) = \delta^{-1} f\left(t\delta^{\frac{1}{k}}\right), \tag{5.18}$$

which has the property that $0 \leq f_\delta(t) \leq 1$ in $|t| \leq K$ and $1 \leq f_\delta^{(k)}(t) = f^{(k)}(t\delta^{1/k}) \leq C$ when $|t| \leq \delta^{-1/k}$. Taylor's formula gives

$$f_\delta(t) = p_{k-1}(t) + r_k(t)\,, \quad p_{k-1}(t) = \sum_{j=0}^{k-1} f_\delta^{(j)}(0)\frac{t^j}{j!}\,,$$

$1/k! \leq |r(t)| \leq C/k!$ when $|t| \leq 1$. Thus, we find that $\sup_{|t|\leq 1} |p_{k-1}(t)| \leq 1 + C/k!$ where $\sup_{|t|\leq 1} |p(t)|$ is a norm on the space of polynomials of degree $k-1$. Since all norms on a finite-dimensional space are equivalent, we find that

$$\sum_{j=0}^{k-1} \left| f_\delta^{(j)}(0)\right| \leq \widetilde{C}_k\,.$$

By using Taylor's formula again, we obtain

$$f_\delta(t) \geq \frac{t^k}{k!} - \widetilde{C}_k \sum_{j=0}^{k-1} \frac{|t|^j}{j!} \gg 1\,, \quad 1 \ll |t| \leq \delta^{-\frac{1}{k}},$$

giving a contradiction for large enough K and small enough δ.

Let ω_δ be given by (5.17), $\chi_\delta(t)$ be the characteristic function of ω_δ, and

$$\Phi_\delta(t) = \delta^{-\frac{1}{k}} \int_{-\infty}^t \chi_\delta(s)ds\,.$$

Then $|\Phi_\delta| \leq C'_k$, so conjugating with $\exp \Phi_\delta(t)$ preserves the L^2 norm and P is transformed into

$$P_\delta = hD_t + i\big(f(t) + h\delta^{-\frac{1}{k}}\chi_\delta(t)\big)\,.$$

Now $f(t) + h\delta^{-1/k}\chi_\delta(t) \geq \min(\delta, h\delta^{-1/k})$, which is optimal when $\delta = h^{k/(k+1)}$. Then we have $P_\delta = hD_t + i(f(t) + h^{k/(k+1)}\chi_\delta(t))$, and since $\frac{1}{2}m(t) \leq f(t) + h^{k/(k+1)}\chi_\delta(t) \leq m(t)$, we obtain

$$\operatorname{Im}\langle P_\delta u, u\rangle \geq \frac{1}{2}\langle mu, u\rangle = \frac{1}{2}\big\|m^{\frac{1}{2}}u\big\|^2 \quad \text{for } u \in \mathcal{C}_c^\infty(\mathbb{R})\,.$$

Since $\operatorname{Im}\langle P_\delta u, u\rangle \leq \|m^{-1/2}P_\delta u\|^2 + \frac{1}{4}\|m^{1/2}u\|^2$, we obtain the result after conjugating back. □

5.3 A Localization Argument

Here we again follow the ideas of [14, sec. 27.3]. To localize the estimate, we take a partition of unity $\{\phi_j(w)\} \in S(1, g_\epsilon)$, where $g_\epsilon = h^{2\epsilon-1}g_h$ and $0 < \epsilon < 1/(2k+2) < \frac{1}{2}$ is fixed; this can be done uniformly in h. Observe that since $2\epsilon < 1/(k+1)$, we find $h^{2\epsilon-1} > h^{-k/k+1} \geq m^{-1}$, so $g_m \leq g_\epsilon$ for any t. We assume that ϕ_j is supported in a sufficiently small g_ϵ neighborhood of w_j, so that $m(t,w) \cong m(t,w_j)$ only varies with a fixed factor in $\operatorname{supp}\phi_j$ when $|t| \leq 1$. Since $\sum \phi_j^2 = 1$ and $g_\epsilon = h^{4\epsilon}g_\epsilon^\sigma$, the calculus gives

$$\begin{aligned}\sum_j \big\|\phi_j^w(x,D_x)u\big\|^2 - Ch^{4\epsilon}\|u\|^2 \leq \|u\|^2 \\ \leq \sum_j \big\|\phi_j^w(x,D_x)u\big\|^2 + Ch^{4\epsilon}\|u\|^2\end{aligned} \tag{5.19}$$

for $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$; thus for small enough h we find

$$\sum_j \big\|\phi_j^w(x,D_x)u\big\|^2 \leq 2\|u\|^2 \leq 4\sum_j \big\|\phi_j^w(x,D_x)u\big\|^2 \quad \text{for } u \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{5.20}$$

We find from Lemma 5.3 that

$$\big\|m_j(t)^{\frac{1}{2}}\phi_j^w u\big\|^2 \leq C\big\|m_j(t)^{-\frac{1}{2}}\phi_j^w(D_t + if_h(t,w_j))u\big\|^2 \quad \forall j \tag{5.21}$$

for $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, where $m_j(t) = m(t,w_j) = f_h(t,w_j) + h^{k/(k+1)}$. The calculus gives

$$\phi_j^w(x,D_x)\big(f_h^w(t,x,D_x) - f_h(t,w_j)\big) = R_j^w(t,x,D_x) \tag{5.22}$$

where $\{m_j(t)^{-1/2}R_j(t,w)\} \in S(h^{1/2-\epsilon}, g_\epsilon)$ when $|t| \leq 1$. In fact, $|f_h(t,w) - f_h(t,w_j)| \leq C\sqrt{m_j}h^{1/2-\epsilon}$ in $\operatorname{supp}\phi_j$ since (5.14) gives that $|f_h|_1^{g_\epsilon} = h^{1/2-\epsilon}|f_h|_1^{g_h} \leq C\sqrt{m_j}h^{1/2-\epsilon}$. Thus, we obtain that

$$\sum_i \big\|m_j(t)^{-\frac{1}{2}}\phi_j^w(x,D_x)\big(f_h^w(t,x,D_x) - f_h(t,w_j)\big)u\big\|^2 \leq Ch^{2\gamma}h^{\frac{k}{k+1}}\|u\|^2 \tag{5.23}$$

where $\gamma = \frac{1}{2} - \epsilon - k/(2k+2) > 0$. Since $h^{k/(k+1)}\|\phi_j^w u\|^2 \leq \|m_j(t)^{1/2}\phi_j^w u\|^2$, we obtain for small enough h from (5.20)–(5.23) that

$$h^{\frac{k}{k+1}}\|u\|^2 \leq C_1 h^{\frac{-k}{k+1}}\|P_h u\|^2 \tag{5.24}$$

for $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ supported where $|t| \leq 1$. This completes the proof of Proposition 5.2. □

6 Dissipative Operators Without the Dynamical Condition

The failure of (1.7) and the consequent failure of Theorem 1.3 are illustrated by the following variant of Davies's example [4] in dimension 2:

$$p(x,\xi) = \xi_1^2 + \xi_2^2 + x_1^2 + i x_2^2 .$$

The spectrum of $p^w(x, hD)$ is given by $\{(2n+1)h\}_{n\in\mathbb{N}} + \{e^{\pi i/4}(2k+1)h\}_{k\in\mathbb{N}}$, and $\Lambda(p)$ is the first quadrant. We see that (1.6) is satisfied everywhere except for $z_0 = 0$. The dynamical condition (1.7) is satisfied on the imaginary half-axis but fails on the real half-axis.

In this section we present a general result in the same spirit. Suppose that $P(h)$ satisfies the general assumptions of Section 2 but does not need to have an analytic symbol. In addition, we assume that

$$P(h) = Q(h) - iW(h)\,, \quad Q(h) = Q(h)^*\,, \quad W(h) \geq 0\,. \tag{6.1}$$

In the classical terminology of [9], this means that our non-self-adjoint operator is dissipative. One way of achieving this semiclassically is by putting

$$Q(h) = q^w(x, hD), \ \ q \text{ real valued}, \quad W(h) = a^{\mathrm{W}}(x, hD) \ \text{ with } a \geq 0\,,$$

$$a^{\mathrm{W}}(x, hD) = \iint a(y,\eta)\Gamma^w_{y,\eta}(x, hD)dy\,d\eta\,, \quad \Gamma_{y,\eta}(x,\xi) = \frac{1}{\pi^n}e^{-|x-y|^2-|\xi-\eta|^2}.$$

Here $\bullet^w$ stands for the Weyl quantization (2.1), and $\bullet^{\mathrm{W}}$ for the Wick quantization. We recall that $a^{\mathrm{W}} = a_0^w$ with

$$a_0(x,\xi) = \iint \Gamma_{y,\eta}(x,\xi)a(y,\eta)dy\,d\eta\,,$$

so that $a \geq 0$ implies $a_0 \geq 0$.

This assumption immediately implies that $\Lambda(p) \subset \{\operatorname{Im} z \leq 0\}$ and also that

$$z \in \sigma(P(h)) \Longrightarrow \operatorname{Im} z \leq 0\,.$$

In fact, we have

$$\begin{gathered} -\operatorname{Im}\langle (P(h)-z)u, u\rangle \geq \operatorname{Im} z\|u\|^2\,, \\ (P(h)-z)^{-1} = \mathcal{O}\left(\frac{1}{\operatorname{Im} z}\right), \quad \operatorname{Im} z > 0\,. \end{gathered} \tag{6.2}$$

We can now use techniques common in the study of dissipative operators; see [9, 11, 19] and references given there. Similar techniques have also been developed in the study of semiclassical resonances [25, 26, 27, 28], and our approach follows these works in an easier setting of dissipative operators.

Thus we start with the following:

LEMMA 6.1 *Assuming* (6.1), *we have, for any open and precompact subset* Ω *of any component of* $\mathbb{C} \setminus \Sigma_\infty(p)$ *intersecting* $\complement\Sigma(p)$ *and for* $0 < g(h) \ll 1$,

$$\|(P(h) - z)^{-1}\| \le \exp\left(C_\Omega h^{-n} \log \frac{1}{g(h)}\right), \quad z \in \Omega \setminus \bigcup_{z_j \in \sigma(P(h))} D(z_j, g(h)) . \tag{6.3}$$

PROOF: We can assume, without loss of generality, that $\Omega = D(z_0, \epsilon)$ and that $D(z_0, 3\epsilon) \cap \Sigma_\infty(p) = \varnothing$. It then follows that for C sufficiently large, $p(x,\xi) \notin D(z_0, 2\epsilon)$ when $|(x,\xi)| > C$. We can then find $p^\# \in \mathcal{C}_\mathrm{b}^\infty(T^*\mathbb{R}^n)$ such that

$$p^\#(x,\xi) = p(x,\xi) \quad \text{for } |(x,\xi)| > C ,$$
$$\forall z \in \Omega,\ \forall (x,\xi) \in T^*\mathbb{R}^n, \quad |(p^\#(x,\xi) - z)^{-1}| \le C .$$

In fact, choose $\alpha : \mathbb{C} \to \mathbb{C} \setminus D(z_0, \epsilon)$ such that $\alpha(w) = w$ on $\mathbb{C} \setminus D(z_0, 2\epsilon)$ and put $p^\# = \alpha \circ p$.

This shows that for h small enough, $p^\#(x, hD) - z$ is invertible, $z \in \Omega$.

In view of the compact support of the symbol $p - p^\#$, we have

$$p(x,hD) - p^\#(x,hD) = A + B , \quad A : L^2(\mathbb{R}^n) \longrightarrow \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n) ,$$
$$B = \mathcal{O}(h^\infty) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) ,$$
$$p(x,hD) - z = \big(p^\#(x,hD) + B - z\big)\big(I + (p^\#(x,hD) + B - z)^{-1} A\big) .$$

Hence $(p^\#(x,hD) - z)^{-1}(p(x,hD) - p^\#(x,hD))$ is a sum of a compact operator on $L^2(\mathbb{R}^n)$ and an operator of small norm.

Since the operator $K(z) = (p^\#(x,hD) + B - z)^{-1} A$ is of trace class, we can define $\det(I + K(z))$, which is holomorphic in Ω. The inverse at a point $z_0 \notin \Sigma(p)$, $(I + K(z_0))^{-1}$, exists for h sufficiently small, and it is bounded independently of h. We also see that

$$|\det(I + K(z_0))|^{-1} \le C_1 \quad \text{with } C_1 \text{ independent of } h.$$

As in Proposition 3.3, $(I + K(z))^{-1}$ is meromorphic and, following [9, chap. 5, theorem 3.1], we see that

$$\|(I + K(z))^{-1}\| \le \frac{\det(I + (K(z)K(z)^*)^{1/2})}{|\det(I + K(z))|} .$$

Hence we need to estimate the determinants from above and below. The upper bound is clear from $|\det(1+A)| \leq \exp \operatorname{tr}(AA^*)^{1/2}$: it is given by $\exp \mathcal{O}(h^{-n})$ since

$$\begin{aligned}\operatorname{tr} \psi(x) b^w(x, hD) \langle hD \rangle^{-m} &= \frac{1}{(2\pi h)^n} \iint \psi(x) b(x,\xi) \langle \xi \rangle^{-m} \, dx \, d\xi \\ &= \mathcal{O}(h^{-n}) \, .\end{aligned}$$

The zeros of $\det(I + K(z))$ coincide with the eigenvalues of $P(h)$, and the usual complex analytic methods (see, for instance, [19, chap. 1] and [25]) show that

$$\begin{gathered}\left| \det(I + K(z)) \right|^{-1} \leq \left(\frac{1}{g(h)} \right)^{Ch^{-n}} \exp(Ch^{-n}) \, , \\ z \in \Omega \setminus \bigcup_{z_j \in \sigma(P(h))} D(z_j, g(h)) \, ,\end{gathered}$$

which proves the lemma. □

The main result of this section will be an application of Lemma 6.1, (6.2), and the following simple function theoretical lemma similar to [28, lemma 2]:

LEMMA 6.2 *Suppose that $F(z)$ is holomorphic in $[-\delta, \delta] + i[-\epsilon, \epsilon]$, and $|F(z)| \leq M$, $M \geq 2$, there. Suppose, in addition, that*

$$|F(z)| \leq \frac{1}{\operatorname{Im} z} \text{ for } \operatorname{Im} z > 0 \quad \text{and} \quad \frac{\epsilon}{\delta} \ll (\log M)^{-1}.$$

Then

$$|F(z)| \leq 2 \frac{\log M}{\epsilon} \text{ for } |z| \leq \frac{\delta}{2} \quad \text{and} \quad \operatorname{Im} z = 0 \, .$$

PROOF: The assumption on $\frac{\delta}{\epsilon}$ allows us to construct a holomorphic function $u(z)$ such that $|u(z)| > \frac{2}{3}$ for $\operatorname{Im} z = 0$, and $|z| < \frac{\delta}{2}$, and $|u(z)| \ll \frac{1}{M}$ for $|\operatorname{Re} z| = \delta$ and $|\operatorname{Im} z| \leq \epsilon$. If we apply an optimized "three-line theorem" argument to $u(z)F(z)$, the lemma follows. □

The two lemmas immediately give the following:

PROPOSITION 6.3 *Suppose that $P(h)$ satisfies (6.1). For $E \in \mathbb{R}$ and $k > 0$, put*

$$\begin{gathered}\Omega(h) = [E - \delta(h), E + \delta(h)] - i \left[0, -\frac{Kh^n \delta(h)}{\log(1/h)} \right], \\ \delta(h) > h^M, \quad K \text{ and } M \text{ large and fixed.}\end{gathered}$$

Then for h small enough, we have

$$\sigma(P(h)) \cap \Omega(h) = \varnothing \Longrightarrow \begin{cases} \|(P(h) - \lambda)^{-1}\| \leq Ch^{-2n} \log^2(\frac{1}{h}) \delta(h)^{-1}, \\ \lambda \in [E - \frac{\delta(h)}{2}, E + \frac{\delta(h)}{2}] \, . \end{cases} \tag{6.4}$$

This means that in small neighborhoods of the real axis, only eigenvalues can produce extreme growth of the resolvent. A contradiction argument from [26, 27, 28] now shows that existence of quasi modes for the operator $P(h)$ implies the existence of spectrum arbitrarily close to the real axis, which, depending on the structure of $W(h)$, can be the boundary of $\Sigma(P)$.

In particular, we have the following:

PROPOSITION 6.4 *Suppose that $P(h)$ satisfies* (6.1) *and that there exists $u(h) \neq 0$ and $\lambda(h) \in \mathbb{R}$ such that*

$$\|(P(h) - \lambda(h))u(h)\| = \mathcal{O}(h^N)\|u(h)\| \quad \textit{for some large } N. \tag{6.5}$$

Then, for h small enough and with K sufficiently large,

$$d(\sigma(P(h)), \lambda(h)) < K h^{N-n} \log\left(\frac{1}{h}\right).$$

This is clear from (6.4), since applying that resolvent estimate to (6.5), we obtain a contradiction.

In the analytic case we expect that if $\operatorname{Im} p$ vanishes to a high order on a closed orbit of $H_{\operatorname{Re} p}$, then we can construct a quasi mode $u(h)$ satisfying (6.5) with a fixed N. If we allow $\mathcal{C}^\infty$ coefficients, then constructing quasi modes with arbitrarily large N's are possible for operators satisfying the assumptions of this section; see [26, 28] and references given there, and also [36, fig. 1] for a figure of an example.

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