Ambiguous Configurations for the 1D Structure and Motion Problem

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Abstract

In this paper we investigate, determine and classify the critical configurations for solving structure and motion problems for 1D retina vision. We give a complete categorization of all ambiguous configurations for a 1D perspective camera irrespective of the number of points and views. Both calibrated and uncalibrated cameras are considered. Several examples and illustrations are provided to explain the results and to provide geometrical insight.

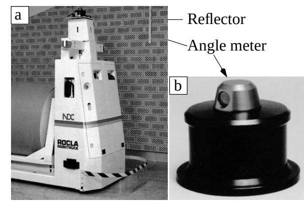

Figure 1. a: A laser guided vehicle. b: A laser scanner or angle meter.

1. Introduction

A key problem in computer vision is to recover the shape of an object from a number of its images. This is known as the *structure and motion problem*. Most work has so far been concentrated on the 2D images of a 3D object [10]. This inverse problem has a number of inherent ambiguities. One well-studied ambiguity is when the visible features lie on a special surface, called a **critical surface**, and the cameras have a certain position relative to the surface. Critical surfaces or "gefährlicher Ort" were already studied by Krames [13] based on a monograph from 1880 on quadrics [17]. In the case of 1D cameras, the situation is less clear. The purpose of this paper is to investigate and classify all critical configurations for the 1D perspective camera.

One-dimensional cameras have proven useful in many applications. In [16] it was shown that the structure and motion problem using lines for affine cameras can be reduced to the structure and motion problem for 1D cameras. Another area of application is vision for planar motion. The ordinary 2D retina vision can be reduced to that of 1D cameras if the motion is planar, i.e. the camera is rotating and translating in one specific plane, cf. [1, 7]. A typical example is the case where a camera is mounted on a vehicle that moves on a flat plane.

The 1D camera may also serve as a good model for the navigation system in *laser guided vehicles* called LGV, see Figure 1. The vision system uses strips of reflector tape which are put on walls or objects along the route of the vehicle, cf. [11]. The *laser scanner* measures the direction from the vehicle to the beacons. This information is used to calculate the position of the vehicle.

In this paper the critical configurations for structure and motion estimation with 1D retina cameras are studied and classified. A complete categorization of the different ambiguities is given, both for calibrated and uncalibrated cameras. The paper is primarily based on three mathematical tools:

- The connection between the calibrated and uncalibrated case through the circular points.
- The Carlsson duality and the Cremona transformation that makes it possible to switch roles between cameras and points.
- Multiview geometry using multilinear constraints. For the 1D retina case there is only the trilinear tensor and its dual (in the Carlsson duality sense).

The approach taken here is much inspired by the works of Carlsson [6], Hartley and Debunne [9] and Maybank [14].

Prior work on critical configurations has been focused on the 2D perspective camera. For critical surfaces and curves, see [13, 5, 14, 8] and for critical camera motions (in the context of auto-calibration), see [19, 12]. In [16], the 1D

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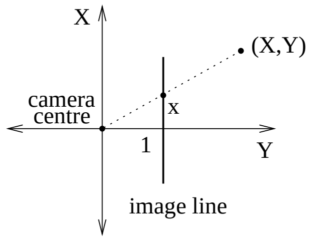

Figure 2. The 1D perspective camera.

camera was studied and it was shown that there are in general two solutions to the structure and motion problem for three views and any number of points. This was further investigated in [3].

The results presented here are of both practical and theoretical interest. On a theoretical level, the 1D retina version of Carlsson duality and the Cremona transformation can be very useful and provide valuable insight into these problems. A complete classification of critical configurations is useful in practical situations when designing measurement paths for structure and motion estimation. Naturally, one wants to avoid the critical configurations.

2. The 1D Perspective Camera

Introduce an object coordinate system and place the camera centre of a perspective camera at the origin and let the camera axis be aligned with the Y-axis, see Fig. 2. Then, a point (X, Y) in the plane is projected to a point x on the image line as $x = \frac{X}{Y}$. Using homogeneous coordinates the projection can be written as a linear equation,

$$\lambda \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}_{\mathbf{U}} \;\Leftrightarrow\; \lambda \mathbf{u} = \mathbf{P}\mathbf{U}, \qquad (1)$$

where λ is a scalar factor, $\mathbf{u}$ and $\mathbf{U}$ are homogeneous coordinate vectors for points on the line and plane, respectively. If the camera centre is at position $\mathbf{C} = (C_x, C_y)$ and the camera axis is not aligned with the Y-axis, then the matrix $\mathbf{P}$ becomes $\mathbf{P} = [\,\mathbf{R}\; -\mathbf{RC}\,]$ where $\mathbf{R}$ is a 2×2 rotation matrix encoding the orientation of the camera.

The 1D perspective camera has also two intrinsic parameters. One is the focal length, which describes the distance from the camera centre to the image line, and the other one is the principal point, which denotes the point on the image line where the camera axis intersects. In the projection equation (1), the focal length is set to one and the principal point to zero. For arbitrary values of focal length f and principal point x_0, the projection is

$$\lambda \mathbf{u} = \begin{bmatrix} f & x_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \end{bmatrix} \mathbf{U} = \mathbf{K} \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \end{bmatrix} \mathbf{U}. \quad (2)$$

When the calibration matrix $\mathbf{K}$ is a priori unknown the camera is said to be **uncalibrated**.

3. Problem Formulation

Motivated by the previous sections the structure and motion problem will now be defined. We formulate it in an uncalibrated camera setting.

Problem 3.1. *Given n image points from m different positions $\mathbf{u}_{I,J}$, $I = 1, \ldots, m, J = 1, \ldots, n$, the* **structure and motion problem** *is to find the depths $\lambda_{I,J} > 0$, the reconstructed points $\mathbf{U}_J$ and the camera matrices $\mathbf{P}_I$ such that*

$$\lambda_{I,J}\mathbf{u}_{I,J} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall I = 1, \ldots, m, J = 1, \ldots, n.$$

We consider two solutions to the structure and motion problem to be the same if they are related by a projective transformation, as they give the same images. Using only two cameras, it is not possible to calculate both structure and motion as any two lines in the plane always intersect. With three measurements of an object point in the plane, there is an addition constraint that the three corresponding lines actually intersect. This can be formulated in the following way, see [16] for a proof.

Theorem 3.1. *Let $\mathbf{u}_{1,J}$, $\mathbf{u}_{2,J}$ and $\mathbf{u}_{3,J}$ be the image points of the same object point from three different camera positions. Then the trilinear constraint*

$$\sum_{i,j,k} T_{ijk} \mathbf{u}^i_{1,J} \mathbf{u}^j_{2,J} \mathbf{u}^k_{3,J} = 0, \qquad (3)$$

is fulfilled for some $2 \times 2 \times 2$ tensor T.

The tensor T encodes the camera motion for three views. Given three camera matrices, it is possible to calculate the corresponding tensor T. The inverse mapping has been studied in [16, 3]. One key result is that given T, there are always two triplets of camera matrices that correspond to T. This implies that for three cameras and any number of points seen in these views, there are always two possible solutions and without any further information, one cannot tell which solutions is the correct one. An additional camera will in general lead to a unique solution.

The two solutions are related by a Cremona transformation [2, 15]. The two solutions coincide in the special case where the three camera centers are aligned. In the special case of calibrated cameras the transformation is well-known to be the so called isogonal conjugacy, cf. [4, p. 113].

4. Uncalibrated vs Calibrated Case

If a camera is calibrated and normalized such that the calibration matrix equals the identity, the first 2×2 matrix of the camera matrix is restricted to a similarity, cf. (1), while an uncalibrated camera matrix is allowed to be a general 2×3 matrix.

Another way to characterize the difference between the calibrated and uncalibrated case is through the use of the circular points, see [18, 3].

Theorem 4.1. *Knowing that the camera is corrected for internal calibration is equivalent to seeing two extra points (the circular points) in each image.*

The important implication of the theorem, that will be used vividly in the sequel, is the following corollary.

Corollary 4.1. *The uncalibrated structure and motion problem with n points and m images is equivalent to the calibrated structure and motion problem with $n-2$ points in m images.*

5. The Carlsson Duality

In [6] Carlsson showed that there is a dual relationship between object points and camera centres for an uncalibrated 2D camera. The Carlsson duality holds also for the case of uncalibrated projections from 2D to 1D.

Following the terminology introduced by Hartley and Debunne [9], the Carlsson duality can be expressed by a certain Cremona tranformation.

Definition 5.1. *The mapping $\Gamma : \mathbf{P}^2 \mapsto \mathbf{P}^2$ given by*

$$(X, Y, Z) \mapsto (YZ, XZ, XY)$$

will be called the Carlsson map. The image of a point $\mathbf{U}$ under Γ will be denoted $\mathbf{U}'$.

The mapping is not defined for any points on the lines joining $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. These three points are called *base points* of the map.

Theorem 5.1 (Carlsson Duality). *The uncalibrated structure and motion problem with n points and m images is equivalent to the uncalibrated structure and motion problem with $m+3$ points and $n-3$ images* [1].

It will be useful to know how the Carlsson map acts on other geometric objects than object points and camera centres. See Page 49, Theorem V in [18] for a proof.

Lemma 5.1. *The Carlsson map transfers (i) a line passing through two general points $\mathbf{U}_1$ and $\mathbf{U}_2$ to a conic through the dual points $\mathbf{U}_1'$, $\mathbf{U}_2'$ and the three base points and (ii) a cubic curve passing through the three base points to a cubic curve passing through the three base points.*

[1] The two problems are not strictly equivalent as the Carlsson map is not bijective on the triangle formed by the base points.

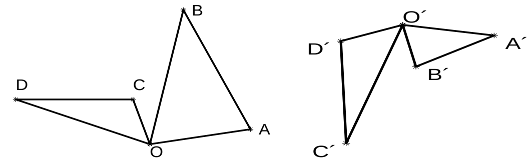

Figure 3. Example of a calibrated Cremona transformation.

In the calibrated case we have two points for "free" (the circular points). These can be used as two of the three base points in forming the Carlsson map. Only one additional point is then needed.

Theorem 5.2. *The calibrated structure and motion problem with n points and m images is equivalent to the calibrated structure and motion problem with $m+1$ points and $n-1$ images.*

The calibrated Cremona transformation can be derived by changing coordinate systems from the one used in the uncalibrated case, to one where the circular points and the origin serve as base points.

Lemma 5.2. *Consider the calibrated Cremona transformation*

$$(X, Y, Z) \mapsto (XZ, -YZ, X^2 + Y^2) \, . \tag{4}$$

The transformation has the property that from every point A the angle measured to an arbitrary point B relative to the origin is the same as the angle from the dual point B' to the point A'.

The lemma is illustrated in Figure 3. Notice that the triangle OAB is congruent to $OB'A'$, the triangles OCD is congruent to $OD'C'$ and similarly for any triangle with O as one of the vertices. This means that if we measure bearings from any set of camera positions $(\mathbf{C}_1, \ldots, \mathbf{C}_m)$ to the the points $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ we will get the same angles as we would if we measured from $(\mathbf{X}_1', \ldots, \mathbf{X}_n')$ to $(\mathbf{C}_1', \ldots, \mathbf{C}_m')$. So an algorithm that solves the structure and motion problem for a particular configuration can also be applied to solve for the dual configuration, cf. [9].

6. Basic Ambiguities

The structure and motion problem can only be determined up to an unknown coordinate transformation. Also, for three cameras and any number of points, there is a two-fold ambiguity. Additionally, there are two other basic ambiguities.

The problem of calculating object points using known camera positions is known as *intersection*. There is one critical configuration for which there is not a unique solution.

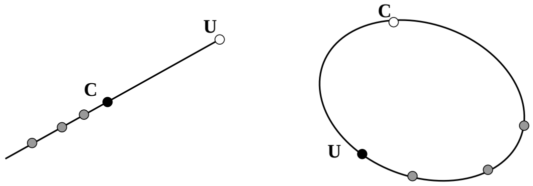

Figure 4. Left: The intersection ambiguity, where a point U and several camera centres C lie on a line. Right: The dual resection ambiguity, where a camera centre C and several points U lie on a conic curve.

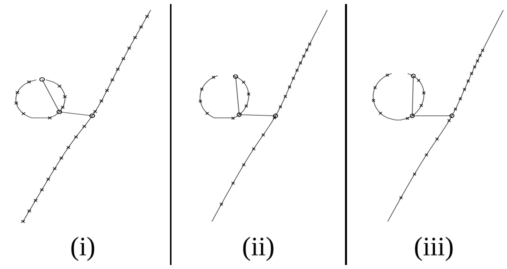

Figure 5. Three cameras (*circles*) are viewing 22 points (*crosses*). All three configurations (out of a one-parameter family) are consistent with the 1D image points. The 25 plane points lie on a cubic.

Theorem 6.1. *Consider the case of several views of one point with known camera matrices. The intersection problem is ambiguous if and only if all camera centres and the point lie on a line.*

Proof. If all camera centres and the object point are aligned, then any point on that line will generate the same image, and therefore ambiguous. If the camera centres are not aligned with the point, then there are lines from (at least) two views which have a unique intersection. □

Resection is the problem of calculating camera positions using image measurements and known object points. In this case, the critical configurations are not obvious. However, the intersection and resection problem are dual to each other, see Figure 4. So by dualizing the above result using the Carlsson map and Lemma 5.1, the following well-known theorem is obtained.

Theorem 6.2. *Consider $m > 4$ object points with known positions and one unknown camera. The resection problem is ambiguous if and only if all points and the camera centre lie on a conic curve.*

7. Three View Ambiguities

A structure and motion problem with three views can be ambiguous in three ways: (i) The alternative reconstructions have the same relative camera motion, (ii) the alternative reconstruction have different relative camera motion, but the corresponding trilinear tensor is the same, or (iii) the alternative reconstructions have different relative camera motion and the corresponding trilinear tensor is different.

For case one there is a unique relative motion, so one can without loss of generality assume that the camera positions are known. The alternative reconstruction differs in at least one of the object points. This can only happen if the camera centres and that point is collinear, see Theorem 6.1. For case two, it is well-known that for each trilinear tensor there are two possible relative camera motions. Thus any three view problem is critical in the sense that there are at least two possible solutions. For the third case, we ask if there are cases where there might be more than two solutions to the structure and motion problem. We will call this case a *three view ambiguity*.

We are now ready to state the theorem describing exactly when there are three view ambiguities. See Figure 5.

Theorem 7.1. *The structure and motion problem for three views and arbitrary number of points is ambiguous if and only if the three camera centres and all the object points lie on a cubic curve.*

There is an interesting special case when all the points and at least one of the camera centres lie on a conic. It fits into the theorem since there is a cubic consisting of the conic through the points and one camera centre and a line through the remaining camera centres. The cubic thus covers all points and camera centres. The problem is then critical in the sense that the resection problem for the first camera is critical, cf. Theorem 6.2.

Proof. Consider a situation where there is an ambiguity. Consider one of the solutions to the problem. For this solution there is a placement of cameras, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. The condition that there is an ambiguous solution is equivalent to saying that there is an alternative tensor T_{ijk} such that

$$\sum T_{ijk}\mathbf{a}^i\mathbf{b}^j\mathbf{c}^k = 0\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are image points in the three images respectively. Since $\mathbf{a}^i = \mathbf{A}^i\mathbf{X}$, $\mathbf{b}^i = \mathbf{B}^i\mathbf{X}$ and $\mathbf{c}^i = \mathbf{C}^i\mathbf{X}$ the constraint on the object point is a third degree polynomial in $\mathbf{X} \in \mathbf{P}^2$:

$$p(\mathbf{X}) = \sum T_{ijk}\mathbf{A}^i\mathbf{X}\mathbf{B}^i\mathbf{X}\mathbf{C}^i\mathbf{X} = 0\,,$$

This shows that all object points pass through this cubic curve. To see that the camera centres lie on the same curve it is sufficient to observe that $\mathbf{AF} = 0$ and thus $p(\mathbf{F}) = 0$, where $\mathbf{F}$ is the camera centre of $\mathbf{A}$.

To show the only if part we consider an object where all camera centres and object points lie on an arbitrary third degree polynomial. Without loss of generality we may change both object coordinate system and image coordinate system so that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as long as the three cameras are not on a line.

The mapping from ambiguous tensors to cubic curves is a linear mapping. Each ambiguous tensor which has eight parameters $T = (T_{111}, T_{112}, T_{121}, T_{122}, T_{211}, T_{212}, T_{221}, T_{222})$ corresponds to a cubic curve

$$p(\mathbf{X}) = \sum T_{ijk} \mathbf{A}^i \mathbf{X} \mathbf{B}^i \mathbf{X} \mathbf{C}^i \mathbf{X} = 0 \ ,$$

where the coefficients $c = (c_{x^3}, c_{x^2y}, c_{x^2z}, c_{xy^2}, c_{xyz}, c_{xz^2}, c_{y^3}, c_{y^2z}, c_{yz^2}, c_{z^3})$ of the polynomial $p(\mathbf{X})$ depend linearly on the tensor coefficients

$$c = MT \,. \tag{5}$$

For this particular choice of coordinates it is straightforward to check that the matrix M has rank 7. If the three camera centres happen to be on a line, it is also easy to check that the corresponding mapping is also linear with rank 7. The mapping (5) is in fact a bijective mapping from the star of tensors through the true tensor (which can be identified with $\mathbf{P}^6$) to the manifold of cubic curves that pass through the three camera centres (also $\mathbf{P}^6$).

Since the mapping is bijective, our arbitrary third degree curve on which the object points lie, correspond to an ambiguous tensor. Thus the structure and motion problem for that case is critical. This concludes the proof. □

From the principle of duality, the following theorem is obtained.

Theorem 7.2. *The structure and motion problem for any number of views of 6 points is ambiguous if and only if the camera centres and the object points lie on a cubic curve.*

Proof. The image under the Carlsson map of a cubic curve through the base points is again a cubic curve through the base point, cf. Lemma 5.1. The dual of 3 cameras and n points is $m = n-3$ cameras and 6 points. So by the principle of duality and Theorem 7.1, the statement is proved. □

8. N View Ambiguities

Up to now, we have limited either the number of cameras or the number of points considered. Based on the previous

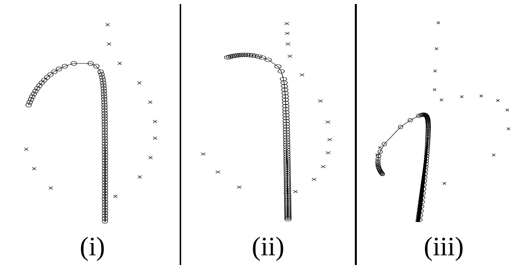

Figure 6. Three solutions (out of a one-parameter family). 82 cameras (*circles*) view 15 points (*crosses*), all lying on a cubic and thus critical.

results, the general problem will now be solved. A natural generalization of the three view case for the word "ambiguous" is that the alternative reconstructions have different relative camera motion and (at least) one triplet of cameras has a different trilinear tensor.

Theorem 8.1. *A 1D structure and motion problem is ambiguous regardless of the number of cameras and points if and only if all the camera centres and the object points lie on a common third degree curve.*

Proof. We begin by showing that a problem is ambiguous if all points lie on a third degree curve. Assume that camera centres and object points lie on a third degree curve c. By first restricting the problem to only 6 points, we know from Theorem 7.2 that the configuration is ambiguous and there is (at least) one-parameter family of solutions. Now, consider a 7th point on the curve c. We need to show that the constraints generated by the projection equation for this extra point does not break the ambiguity. However, all these constraints reduce to trilinear constraints, as there are no higher order constraints for 1D camera motion, cf. [3]. Thus, it suffices to consider three arbitrary cameras $\mathbf{P}_i$, $\mathbf{P}_j$ and $\mathbf{P}_k$. In the proof Theorem 7.1, we showed that the map from stars of tensors to cubic curves (through the camera centres) is bijective. So, from c and the three cameras centres, a star of tensors $\lambda T_1 + \mu T_2$, where $(\lambda, \mu) \in \mathbf{P}^1$, is obtained. But according to the proof of Theorem 7.1, as long as the 7th point is on c, all tensors in $\lambda T_1 + \mu T_2$ are still valid solutions.

To show that each ambiguous problem has the property that all points lie on a third degree curve we use a proof by contradiction. Assume, thus that there exist ambiguous problems with m views of n points such that the $m+n$ points do not lie on a common third degree curve. Such problems must have $m > 3$ and $n > 6$ because of Theorems 7.1 and 7.2 respectively. Study such a problem where $m+n$ is minimal. If we remove one point or one camera, we obtain an ambiguous problem with one point less. By the assumption these $m+n-1$ points must lie on a third

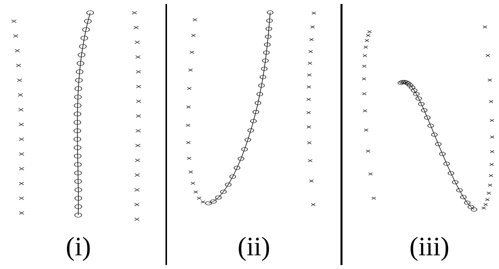

Figure 7. Three solutions (out of a one-parameter family). The camera moves in a corridor with scene points on both walls, which is frequently occurring in practice. 25 cameras view 29 points, all lying on a cubic.

degree curve. In particular, this means that all 10 point sub-configurations must lie on a third degree curve. According to Lemma A.1, all $m+n$ points then lie on a cubic curve. Thus all ambiguous configurations have the property that all $m+n$ points lie on a third degree curve. □

In Figure 6 an example of a critical configuration is illustrated. Even though there are 82 views of 15 points, the 1D images alone cannot disambiguate between a one-parameter family of solutions. Another example is illustrated in Figure 7, where a camera moves along corridor which is frequently occurring in practical situations.

9. Conclusions

We have given a complete categorization of all ambiguous configurations for the structure and motion problem in 1D retina vision. The main ambiguity is when all object points (regardless of how many) and all camera centres (again, regardless of the number of cameras) lie on a cubic.

Appendix

Lemma A.1. *Let $\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_N$ denote $N > 10$ arbitrary points in $\mathbf{P}^2$. If, for each combination of 10 points, there exists a cubic curve through these 10 points, then there exists a cubic curve through all N points.*

Proof. For each point $\mathbf{Q}_i = (X_i, Y_i, Z_i)^T$, let $\tilde{\mathbf{Q}}_i = (X_i^3, X_i^2 Y_i, ..., Z_i^3)^T$, i.e. a 10 vector containing all cubic monomials. Then, $\mathbf{Q}_i$ lies on a cubic with coefficients $c = (c_{x^3}, c_{x^2y}, ..., c_{z^3})^T$ if and only if $c^T \tilde{\mathbf{Q}}_i = 0$. Further, let $M = \begin{bmatrix} \tilde{\mathbf{Q}}_1 & \tilde{\mathbf{Q}}_2 & ... & \tilde{\mathbf{Q}}_N \end{bmatrix}$, which is a $10 \times N$ matrix. As each combination of 10 points lies on a cubic, it follows that each 10×10 submatrix of M has a non-empty left nullspace. This implies that $\operatorname{rank} M \leq 9$ and there is non-empty left nullspace of M. Let c_M be a vector in that nullspace. Thus $c_M^T \tilde{\mathbf{Q}}_i = 0$ and all points lie on the cubic corresponding to c_M. □

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