# Answers and Hints

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## 1 Points and Vectors

1. New coordinates of the point $\begin{bmatrix} x \\ y \end{bmatrix}$ are $\begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$. The following table

| | $P$ | $Q$ | $P+Q$ | $\frac{P+Q}{2}$ |
|---|---|---|---|---|
| *old* | $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ | $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ |
| *new* | $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ | $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ | $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ |

shows that in new coordinates $P+Q$ be another point than $\begin{bmatrix} 2+1 \\ 4+1 \end{bmatrix}$, but $\frac{P+Q}{2}$ be the same as $\begin{bmatrix} 1+1 \\ 2+1 \end{bmatrix}$.

2. $P = \frac{3}{5}P_0 + \frac{2}{5}P_1$ and coordinates are $(\frac{3}{5}, \frac{2}{5})$.

3. $2 = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 5$. Similarly

$$1 = \frac{4}{5} \cdot 0 + \frac{1}{5} \cdot 5 \Rightarrow 0 = \frac{5}{4} \cdot 1 - \frac{1}{4} \cdot 5.$$

Answer: $(\frac{3}{4}, \frac{1}{4}), (\frac{5}{4}, \frac{-1}{4})$.

4. If $P_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then for arbitrary point

$$\begin{bmatrix} x \\ y \end{bmatrix} = (1-x-y)P_0 + xP_1 + yP_2.$$

5.

$$\sum_{i=0}^{n} \alpha_i P_i = \sum_{i=0}^{n} \alpha_i P_i - \left( \sum_{i=0}^{n} \alpha_i \right) P_0 = \sum_{i=0}^{n} \alpha_i (P_i - P_0)$$

is a vector.

6. Let $P = \begin{bmatrix} x \\ y \end{bmatrix} = x_0 P_0 + x_1 P_1 + x_2 P_2$. Then $0 \leq x_i$ if $P$ is inside the triangle $P_0 P_1 P_2$, $x_0 = 0$ means that $P$ is on the line $P_1 P_2$, $x_0 < 0$ means that $P$ is on the other side of the line $P_1 P_2$ than the triangle. Combining signs of $x_i$ we can describe all regions. Note that not all of them are negative, because their sum is equal to 1.

# 2 Mass-center (center of gravity)

1. According to the Theorem 2 the point $C_1$ can be written as $C_1 = \frac{3}{5}A + \frac{2}{5}B$ and point $O$ can be written as $O = \frac{10}{10+3}C_1 + \frac{3}{10+3}C$. Together we get

$$O = \frac{10}{13} \left( \frac{3}{5}A + \frac{2}{5}B \right) + \frac{3}{13}C = \frac{6}{13}A + \frac{4}{13}B + \frac{3}{13}C,$$

which is the same result.

2. See Theorem 9 later.

3. See Theorem 9 later.

4. Let $\alpha, \beta$ and $\gamma$ be angles in the triangle. The altitude from the angle $\gamma$ divides the opposite side in the relation $\cot \alpha : \cot \beta$ (with a natural interpretation of the negative values). Ceva theorem, generalized to the negative values, shows that the altitudes have the common point $O$, which has the coordinates

$$\left( \frac{\cot \beta \cdot cot\gamma}{\cot \alpha \cdot \cot \beta + \cot \alpha \cdot cot\gamma + \cot \beta \cdot cot\gamma}, \right.$$

$$\frac{\cot \alpha \cdot cot\gamma}{\cot \alpha \cdot \cot \beta + \cot \alpha \cdot cot\gamma + \cot \beta \cdot cot\gamma},$$

$$\left. \frac{\cot \beta \cdot cot\alpha}{\cot \alpha \cdot \cot \beta + \cot \alpha \cdot cot\gamma + \cot \beta \cdot cot\gamma} \right).$$

# 3 Affine maps

1.

$$\Phi \left( \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \Phi \left( \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \Phi \left( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ -2 \end{bmatrix},$$

$$\Phi\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ -6 \end{bmatrix}, \Phi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -5 \end{bmatrix}, \Phi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -5 \end{bmatrix},$$

$$\Phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

2. Affine, nonlinear (the origin moves).

$$\Phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3. Neither linear nor affine.

$$\Phi\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5/2 \\ 0 \end{bmatrix}, \Phi\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 5/3 \end{bmatrix},$$

$$\Phi\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \frac{1}{2}\Phi\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) + \frac{1}{2}\Phi\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5/4 \\ 5/6 \end{bmatrix}.$$

# 4 The Casteljau Algorithm and Bézie curve

1.

$$b(t) = \begin{bmatrix} 2t^2 + 2t \\ 6t - 6t^2 \end{bmatrix}.$$

2. If $P$ and $Q$ are on the line $l$ then all the points $(1-t)P + tQ$ belongs to $l$, and all the points in the Casteljau algorithm belongs to $l$. That is why $b(t) \in l$. To prove the inverse is more difficult. The shortest way is to use the derivative (which follows later). On this level it is possible to do as follows. First prove that if $b(t) = O$, then all points $P_i$ coincides with the origin $O$. Second, use the variable change to get the line as $x-$axes. Then forget the x-coordinate and use the previous statement about the origin.

3.

$$P_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, P_1 = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, P_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. If $P$ and $Q$ are inside the triangle, then all the segment $(1-t)P + tQ$ for $0 \le t \le 1$ is inside the triangle. That is why all the points during calculations in the Casteljau algorithm (including the last one $b(t)$) are inside the triangle.

5.

$$b(t) = \begin{bmatrix} 6t - 6t^2 \\ 6t^2 - 4t^3 \end{bmatrix}.$$

# 5 Induction

1. Induction step $n \Rightarrow n+1$. We need to prove that

$$1^2 + 2^2 + \dots n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

By the induction hypothesis it is equivalent to

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Cancelling nonzero terms we get

$$n(2n+1) + 6(n+1) = (n+2)(2n+3),$$

that can be checked directly.

2. One nee only to guess that

$$1^3 + 2^3 + \dots n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

and use the same type of a prove as above.

3. It is easier to draw starting with the triangle with a large $x-$coordinate.

4. $\begin{bmatrix} 5/4 \\ 3/2 \end{bmatrix}$ and $\begin{bmatrix} 1/3 \\ 2 \end{bmatrix}$. Negative mass lead us outside the triangle.

# 6 Binomial coefficients

1. Take $n$ points on the line and put somewhere between them $k-1$ separators. Every choice of the separators gives a solution. Now mix separators and points. We get $n+k-1$ objects from which we need to choose $k-1$ separators.

2. Consider $0 = (1-1)^n$.

3. For the second prove consider $(1+1)^n$. For the last note that the total number of subsets is equal to $2^n$ and $\binom{n}{k}$ is equal to the number of subsets consisting of $k$ elements.

4. Without the induction. To choose the first element we have $n$ possibilities, to choose the second $n-1$ and so on. The result is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Because the ordering is not important some choices gave the same result and we should divide this by $k!$ to get the correct formula.

## 7 Bernstein polynomials

1. 
$$b(t) = \begin{bmatrix} 6t^2 - 4t^3 \\ 6t - 6t^2 \end{bmatrix}.$$

2. Coefficient with $t^j = t^i t^{j-i}$ is equal to

$$\binom{n}{i}(-1)^{j-i}\binom{n-i}{j-i} = (-1)^{j-i}\frac{n!}{i!(n-i)!}\cdot\frac{(n-i)!}{(j-i)!(n-j)!} =$$

$$(-1)^{j-i}\frac{n!}{(n-j)!}\cdot\frac{1}{(j-i)!i!} = (-1)^{j-i}\frac{n!}{(n-j)!j!}\cdot\frac{j!}{(j-i)!i!}.$$

3. $b(t)$ is obtained from $n-k+1$ points $b_i^k(t)$ using the Casteljau algorithm.

4. Check using induction and the Casteljau algorithm that $b_i^k(t) = \frac{i+kt}{n}$.

5. Note that $B_i^n(1-t) = B_{n-i}^n(t)$.

6. Write explicit all binomial coefficients and cancel factorials.

7. Use the exercise 2 from this and the previous section.

## 8 Properties of Bézie curve

1. See the section 17.

2. 

| $i=$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $n=3$ | 1 | 4/9 | 4/9 | 1 | – | – |
| $n=4$ | 1 | 27/64 | 3/8 | 27/64 | 1 | – |
| $n=5$ | 1 | 256/625 | 216/625 | 216/625 | 256/625 | 1 |

3. $b(t) = \begin{bmatrix} 4t \\ 8t \end{bmatrix}.$

4. $-2 \le x \le 0, \ -3 \le y \le 3.$

$$b(t) = \begin{bmatrix} 3t^2 - 4t \\ -9t^2 + 6t \end{bmatrix}.$$

## 9 Parameter transformations

1. We still have $y = x - x^2/2$ in

$$\begin{bmatrix} \frac{2(u+2)}{3} \\ \frac{2(u+2)}{3} - \frac{2(u+2)^2}{9} \end{bmatrix}.$$

2.

$$t = \frac{u+0}{a-0} = \frac{u}{a} \Rightarrow u = at.$$

$$c(u) = b(\frac{u}{a}) \Rightarrow c(at) = b(t).$$

3.

$$t = \frac{u-\alpha}{\beta-\alpha} \Rightarrow u = (\beta-\alpha)t + \alpha \Rightarrow$$

$$b(t) = c((\beta-\alpha)t + \alpha).$$

## 10 Subdivision

1.

$$Q_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}, Q_2 = \begin{bmatrix} 4/3 \\ 4/9 \end{bmatrix}.$$

2.

$$Q_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}.$$

$$Q_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}, Q_2 = \begin{bmatrix} 1/2 \\ 3/8 \end{bmatrix}.$$

3.

$$Q_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, Q_2 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$

$$Q_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}.$$

## 11 Difference operator

1.

$$\Delta^4 a_n = \Delta(a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n) =$$

$$a_{n+4} - a_{n+3} - 3(a_{n+3} - a_{n+2}) + 3(a_{n+2} - a_{n+1}) - a_{n+1} + a_n =$$

$$a_{n+4} - 4a_{n+3} + 6a_{n+2} - 4a_{n+1} + a_n.$$

2.

$$\Delta^{k+1} a_n = \Delta \left( \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a_{n+j} \right) =$$

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \Delta(a_{n+j}) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (a_{n+j+1} - a_{n+j}) =$$

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}a_{n+j+1} - \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}a_{n+j} =$$

$$\sum_{j'=1}^{k+1}\binom{k}{j'-1}(-1)^{k-j'+1}a_{n+j'} - \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}a_{n+j} =$$

$$\sum_{j=1}^{k+1}\binom{k}{j-1}(-1)^{k-j+1}a_{n+j} + \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j+1}a_{n+j} =$$

$$\sum_{j=0}^{k+1}\binom{k}{j-1}(-1)^{k-j+1}a_{n+j} + \sum_{j=0}^{k+1}\binom{k}{j}(-1)^{k-j+1}a_{n+j} =$$

$$\sum_{j=0}^{k+1}\left[\binom{k}{j-1} + \binom{k}{j}\right](-1)^{k-j+1}a_{n+j} =$$

$$\sum_{j=0}^{k+1}\binom{k+1}{j}(-1)^{(k+1)-j}a_{n+j}.$$

3. Note that if $f(t) = t^m$ then $\Delta f(n) = (n+1)^m - n^m$ is a polynomial of the degree $m-1$ according to Theorem 10. It remains to use the induction by $k$.

4. $a_n = n^3 - n + 1$ if we start with $a_0$.

5. $a_n = 3a_{n-1} - a_{n-2}, n > 2$.

# 12 The derivative of Bézie curve

1.
$$\frac{d^{k+1}}{(dt)^{k+1}}b(t) = \frac{d^k}{(dt)^k}\left(n\sum_{i=0}^{n-1}B_i^{n-1}(t)\Delta P_i\right) =$$

(using induction hypothesis to $\sum_{i=0}^{n-1}B_i^{n-1}(t)\Delta P_i$, and $(n-1)+1$ "points" $\Delta P_i, i = 0, 1\ldots, n-1$)

$$n\left((n-1)(n-2)\cdots(n-1-k+1)\sum_{i=0}^{n-1-k}B_i^{n-1-k}(t)\Delta^k(\Delta P_i)\right) =$$

$$n(n-1)(n-2)\cdots(n-(k+1)+1)\sum_{i=0}^{n-(k+1)}B_i^{n-(k+1)}(t)\Delta^{k+1}P_i.$$

2.
$$(1-t)^2\begin{bmatrix}0\\0\end{bmatrix} + 2t(1-t)\begin{bmatrix}1/2\\0\end{bmatrix} + t^2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}t\\t^2\end{bmatrix}.$$

3.

$$P_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, P_1 = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1/2 \\ 1/6 \end{bmatrix}, P_3 = \begin{bmatrix} 3/4 \\ 1/2 \end{bmatrix}, P_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. Theorem 21 means that all points can be obtained from the derivatives in zero.

5. use the fact that $B_i^k(1) = 0$ if $i \neq k$.

# 13 Degree elevation

1.

$$R_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, R_1 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, R_2 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}, R_3 = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}, R_4 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

2. $P_i = \begin{bmatrix} i/n \\ 2i/n \\ 3i/n \end{bmatrix}, i = 0, 1 \ldots, n.$

3. For $k = 1$ we have only two non-zero summands when $i - j = 0$ and $i - j = 1 \Leftrightarrow j = i - 1$ :

$$Q_i^1 = \binom{n}{i} \frac{\binom{1}{0}}{\binom{n+1}{i}} P_i + \binom{n}{i-1} \frac{\binom{1}{1}}{\binom{n+1}{i}} P_{i-1} =$$

$$\frac{n!}{i!(n-i)!} \cdot \frac{i!(n+1-i)!}{(n+1)!} P_i + \frac{n!}{(i-1)!(n-i+1)!} \cdot \frac{i!(n+1-i)!}{(n+1)!} P_{i-1} =$$

$$\frac{n+1-i}{n+1} P_i + \frac{i}{n+1} P_{i-1}.$$

# 14 Nonparametric Curves and Integrals

1. $b_0 = 0, b_1 = \frac{1}{3}, b_2 = \frac{2}{3}, b_3 = 0.$

2. $\frac{1}{4}\left(0 + \frac{1}{3} + \frac{2}{3} + 0\right) = \frac{1}{4}.$

3. We get an interval between 0 and 1 on $x-$axis:

$$b(t) = \begin{bmatrix} \frac{t+t^2}{2} \\ 0 \end{bmatrix}.$$

4. $\frac{ai}{n} + b, i = 0, 1 \ldots, n.$

## 15 Lagrange interpolation

1.
$$\begin{bmatrix} t^2 - t \\ t + 2 \end{bmatrix}.$$

2. Note that $P(x)$ have only even powers and can be written as $P(x) = Q(t)$, where $t = x^2$ and $Q(1) = 5, Q(2) = 12, Q(3) = 4, Q(4) = 6$. The answer is $P(0)Q(0) = -42$.

3. It is a segment of a line.

## 16 The Newton Form of Interpolation

1.
$$P(t) = \begin{bmatrix} t^2 - t \\ \frac{-t^3+3t^2+73t+21}{48} \end{bmatrix}, P(9) = \begin{bmatrix} 72 \\ 4 \end{bmatrix}.$$

2. Note that
$$\frac{t(t-1)(t-2)\cdots(t-k+1)}{k!} = \binom{t}{k}.$$

3.
$$P_0+\binom{n+1}{1}\Delta P_0+\binom{n+1}{2}\Delta^2 P_0+\binom{n+1}{3}\Delta^3 P_0+\cdots+\binom{n+1}{n+1}\Delta^{n+1} P_0 =$$
$$P_0+\left(\binom{n}{0}+\binom{n}{1}\right)\Delta P_0+\left(\binom{n}{1}+\binom{n}{2}\right)\Delta^2 P_0+\left(\binom{n}{2}+\binom{n}{3}\right)\Delta^3 P_0+\cdots+$$
$$\left(\binom{n}{n-1}+\binom{n}{n}\right)\Delta^n P_0\left(\binom{n}{n}+0\right)\Delta^{n+1} P_0 =$$
$$P_0+\binom{n}{1}\Delta P_0+\binom{n}{2}\Delta^2 P_0+\binom{n}{3}\Delta^3 P_0+\cdots+\binom{n}{n}\Delta^n P_0+$$
$$+\Delta P_0+\binom{n}{1}\Delta^2 P_0+\binom{n}{2}\Delta^3 P_0+\binom{n}{3}\Delta^4 P_0+\cdots+\binom{n}{n}\Delta^{n+1} P_0 =$$
$$P_n + \Delta P_n = P_n + P_{n+1} - P_n = P_{n+1}.$$

4. Prove it first for $h = 1$ and $t_0 = 0$.

## 17 Symmetry and Extrapolation

1. We use induction by $k$ to show that $\Delta^k P_i$ corresponds to $(-1)^k \Delta^k P_{n-k-i}$. For $k = 0$ it is line 3.

$$\Delta^{k+1} P_i = \Delta^k \Delta P_i = \Delta^k (P_{i+1} - P_i) = \Delta^k P_{i+1} - \Delta^k P_i$$

corresponds by induction to

$$(-1)^k \Delta^k P_{n-k-i-1} - (-1)^k \Delta^k P_{n-k-i} = -(-1)^k \Delta^k (P_{n-k-i} - P_{n-k-i-1}) =$$

$$(-1)^{k+1} \Delta^k \Delta P_{n-k-i-1} = (-1)^{k+1} \Delta^{k+1} \Delta P_{n-(k+1)-i}.$$

2.

$$Q_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, Q_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, Q_2 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 28 \\ -8 \end{bmatrix}.$$

3. Use the extrapolation procedure with $d = \frac{1}{2}$ but change the order of the points to opposite ones, so

$$Q_0 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, Q_1 = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}, Q_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

4. Small change leads to unexpected large difference in results.

## 18 Spline Curves. Global and local parameters

1.

$$s(u) = \begin{cases} \begin{bmatrix} 2u \\ 2u - u^2 \end{bmatrix}, & \textit{if} \quad 0 \le u \le 1 \\ \begin{bmatrix} u + 1 \\ -u^2/4 + u/2 + 3/4 \end{bmatrix}, & \textit{if} \quad 1 \le u \le 3 \end{cases}.$$

Continuous, but is not smooth (the derivatives in 1 are different).

2. It is a triangle.
3. Note that $\frac{dt}{du} = \frac{1}{\Delta_i}$ and $\frac{ds}{du} = \frac{ds}{dt} \cdot \frac{dt}{du}$.
4. Bézie curve $b(t)$ is a closed curve: $b(0) = b(1)$.
5. Yes.

## 19 Smoothness Conditions

## 20 Conditions $C^1$ and quadratic B-splines

1.
$$P_2 = \frac{2}{3}\begin{bmatrix} 3 \\ 3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$
$$s(u) = \begin{cases} \begin{bmatrix} 6u - 2u^2 \\ 6u - 4u^2 \end{bmatrix}, & \textit{if} \quad 0 \le u \le 1 \\ \begin{bmatrix} u^2/4 + 3u/2 + 9/4 \\ 5u^2/4 - 9u/2 + 21/4 \end{bmatrix}, & \textit{if} \quad 1 \le u \le 3 \end{cases}.$$

2. It is impossible because $P_1, P_2$ and $P_3$ are not on the same line. If we replace $P_2$ by $\begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$ then it is possible and $u_1 = 1$.

3. Every point $P_{2i}$ is inside, so every triangle $P_{2i}P_{2i+1}P_{2i+2}$ is inside. But the corresponding Bézie curve is inside the triangle.

4. Use $\Delta_{L-1} : \Delta_0$.

## 21 Finding a Knot sequence. Font Design.

1. $a, = b = 5$,
$$u_0 = 0, u_1 = \frac{6}{7}, u_2 = \frac{18}{7}, u_3 = 6.$$

## 22 $C^2-$Conditions and Cubic Splines

1. No, $C^1$ only.

2. $L + 3$.

## 23 The Blossom

1.
$$b[t_1, t_2, t_3] = (1-t_1)(1-t_2)(1-t_3)P_0 + (t_1(1-t_2)(1-t_3) + [1-t_1)t_2(1-t_3) + (1-t_1)(1-t_2)t_3]P_1 +$$
$$[t_1t_2(1-t_3) + t_1(1-t_2)t_3 + (1-t_1)t_2t_3]P_2 + t_1t_2t_3]P_3.$$

## 24 Bézie surfaces

1.
$$\begin{bmatrix} -u^2v^2 + 4uv^2 + v^2 + 2u - 4uv \\ u^2v^2 - v^2 + 2v \\ v^2 + 4uv - 4uv^2 \end{bmatrix}$$

2. Note that if two are on the plane then every point on the segment between them is this plane too.

## 25 The Tensor Product Approach

1.
$$b^{2,1}(u,v) = \begin{bmatrix} 2u - 2uv \\ v \\ 2uv \end{bmatrix}.$$

2.
$$P_0 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}, P_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, P_1 = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}.$$

3. Isoparametric curves are those lines.

## 26 Degree Elevation and Derivatives

1.
$$P_{0,0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, P_{1,0} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}, P_{2,0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$P_{0,1} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}, P_{1,1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/4 \end{bmatrix}, P_{2,1} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix},$$
$$P_{0,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, P_{1,2} = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}, P_{2,2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

2.
$$\frac{\partial b}{\partial u}(1/2,1/2) = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}, \frac{\partial b}{\partial v}(1/2,1/2) = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}.$$