You may not consult with any other person regarding the exam, but you may use books, software, internet and other sources of information. Use, when necessary, MAPLE or any computer algebra system, but be able to explain algorithms in the cases where Maple solves the problem directly. All varieties in the problems (if not specified) are over the complex numbers. Each problem gives up to 1.0 point. To pass the exam you need at least 2 points for the first four problems and at least 0.5 points for each of two last problems. Send your Maple/Matlab calculations as separate files by e-mail to ufn@maths.lth.se. You may be asked to take part in an oral discussion to check or complete your solutions.

1. Identify the set of pairs \((f(x), g(x))\) of polynomials of degree less than 3 with \(\mathbb{C}^6\). Let \(X\) be a subset of those \((f, g)\) for which \(f(x)\) is monic. Which of the following subsets in \(\mathbb{C}^6\) and in \(X\) are affine varieties:
   - Those \((f, g)\) for which \(f(x)g(x)\) is an even function.
   - Those \((f, g)\) for which \(f(x) \neq g(2x)\) for all \(x\).
   - Those \((f, g)\) for which \(g(x)\) is monic and \(f(x)\) and \(g(x)\) have a common root.
   - Those \((f, g)\) for which \(fg = x^3 + 4x\).
   - Those \((f, g)\) for which \(g\) is monic, has degree 2 and divides \(f\).

2. Let \(A, B, C\) be three different points in a real plane. Prove the following geometric statements using Gröbner basis methods.
   a) The altitudes \(AA_1, BB_1, CC_1\) (where \(A_1\) belongs to the line \(BC\), \(B_1\) belongs to the line \(AC\) and \(C_1\) belongs to the line \(AB\)) have a common point \(H\). **Hint:** for all points \((x_i, y_i)\) named here write the condition as polynomial equations. Show that the system has non-trivial real solutions (trivial solutions we get when two of points \(A, B, C\) coincide). The calculations will be easier if we first choose the coordinates such that \(A = (0, 0), B = (1, 0)\).
   b) Show that the midpoints of the segments \(AB, AC, BC, AH, BH, CH\) belong to the circle through \(A_1, B_1, C_1\). **Hint:** One possible way is to use new variables \(a, b, R\) to describe the circle’s equation and study the corresponding system of equations.

3. Let \(a_1, \ldots, a_n\) be \(n\) different complex numbers. It is known that the affine variety defined by the following \(n\) equations: \(\sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} a_i^k, k = 1, \ldots, n\) consists of all points having different \(a_j\) as different coordinates (i.e. points \((a_{\sigma(1)}, \ldots, a_{\sigma(n)})\) for any permutation \(\sigma\)).
   a) Show this for small \(n = 2, 3, 4\) (for the last one you probably need a Gröbner basis calculation).
b) A $3 \times 3$ magic square contains 9 different numbers $1, \ldots, 9$ such that the sums in every row, column and two main diagonals are the same. Use the result above to describe the set of magic squares by equations. Find the Gröbner basis for the corresponding set of polynomials. Use it to find an example of such square and the number of such squares.

4. a) Let $X$ be an affine variety and $U_i$ is an infinite set of (Zariski) open sets such that $X \subseteq \bigcup_i U_i$. Show that there exists a finite subset $U_{i_1}, \ldots, U_{i_k}$ such that $X \subseteq \bigcup_{j=1}^k U_{i_j}$.

b) Let $X, Y \subseteq \mathbb{C}^n$ be an affine varieties and $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a regular map. Which of the following sets are an affine:

i) $f(X) = \{ f(x) | x \in X \}$?

ii) $f^{-1}(Y) = \{ x | f(x) \in Y \}$?

iii) $\{(x, y) \in \mathbb{C}^{2n} | x \in X, y \in Y \}$?

iv) $\{(x, f(x)) \in \mathbb{C}^{2n} | x \in X \}$?

5. Assume that we are given two polynomials $f_1$ and $f_2$, with complex coefficients, both containing all the monomials from

$$M = \{ x^6 y^3, x^5 y^3, x^4 y^3, x^4 y^2, x^3 y^3, x^2 y^3, x^2 y^2, x^2 y, x y^3, x y^2, x y, y^3, y^2, y, 1 \}.$$  

We would like to find out how many solutions, $(x, y) \in (\mathbb{C}^*)^2$, we get to

$$f_1 = f_2 = 0. \tag{1}$$

a) Construct the convex polytopes $P_1$ and $P_2$ corresponding to $f_1$ and $f_2$.

b) What is the Minkowski sum $P = P_1 + P_2$?

c) Construct all possible mixed subdivisions of $P$.

d) For each mixed subdivision, $R_1, \ldots, R_s$ of $P$, which are the mixed cells $R_i$? How many solutions do we get to (1), for generic choices of coefficients?

6. Consider the ideal

$$I = \langle x^2 y - y^3 - 3x^2 + 3y^2, x^3 - x y^2 - x^2 + y^2, y^4 + 9x^2 - 10y^2, x y^3 - x y^2 - y^3 + y^2 \rangle.$$  

a) Find a set of monomials that constitute a basis of $A = \mathbb{C}[x, y]/I$.

b) Find $I \cap \mathbb{C}[y]$ and use this to find $V(I)$. Compare $|V(I)|$ and $\dim(A)$. What does this tell us about the ideal $I$?

c) Use the eigenvalue method to find $V(I)$.

d) Compute $\sqrt{I}$ and use the eigenvalue method on $\sqrt{I}$ to find $V(I)$.

\[1\) This problem occurs when one wants to solve the so-called image stitching problem in computer vision, for a purely rotating camera, with unknown focal length and unknown radial distortion.