On Random Bernoulli Convolutions

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With one figure

Abstract

We study the distribution of the random series \( \sum_{k=0}^{\infty} \pm \lambda_k^k \) where \( \lambda_k \) are independently and uniformly distributed in \((\lambda - \epsilon, \lambda + \epsilon)\). It is proved that the distribution of the series has density in \( L_2 \) and that the \( L_2 \) norm of the density does not grow faster than \( 1/\sqrt{\epsilon} \).

1 Introduction and Statements of Results

Let \( \lambda \in (\frac{1}{2}, 1) \). The distribution of the random variable \( Y_\lambda \), defined by

\[
Y_\lambda = (1 - \lambda) \sum_{k=0}^{\infty} \vartheta_k \lambda^k,
\]

where the random variables \( \vartheta_k \) are independent and identically distributed according to \( P(\vartheta_k = +1) = P(\vartheta_k = -1) = 1/2 \), is denoted by \( \nu_\lambda \). We have chosen the constant \((1 - \lambda)\) in front of the sum in order to have \( \nu_\lambda \) with the support \([-1, 1]\) for all \( \lambda \). This distribution was studied by Erdös in [1] and [2]. It is known that \( \nu_\lambda \) is absolutely continuous with respect to Lebesgue measure for almost all \( \lambda \in (\frac{1}{2}, 1) \), see [7] and [4], and that if \( \lambda^{-1} \) is a Pisot number, then \( \nu_\lambda \) is singular with respect to Lebesgue measure, [1]. It is also known that for \( \lambda = 1/\sqrt{7} \), \( k \in \mathbb{N} \), and for some algebraic integers, \( \nu_\lambda \) is absolutely continuous with respect to Lebesgue measure, [3], [9]. There is a nice survey of the subject in [5].

For \( \varepsilon > 0 \), we study the distribution of the random variable

\[
Y_{\lambda, \varepsilon} = (1 - \lambda - \varepsilon) \sum_{k=0}^{\infty} \vartheta_k \lambda^k,
\]

where the random variables \( \lambda_k \) are uniformly and independently distributed in \((\lambda - \varepsilon, \lambda + \varepsilon)\), and the random variables \( \vartheta_k \) are independent, independent of \( \lambda_k \) and distributed as above. We let \( \nu_{\lambda, \varepsilon} \) denote the distribution of the random variable \( Y_{\lambda, \varepsilon} \). The coefficient \( 1 - \lambda - \varepsilon \) in the definition of \( Y_{\lambda, \varepsilon} \) is chosen in order that the measure \( \nu_{\lambda, \varepsilon} \) has support in \([-1, 1]\).
It is not hard to see that the distribution $\nu_{\lambda, \varepsilon}$ approximates $\nu_{\lambda}$. More precisely, one easily proves the following theorem.

**Theorem 1.** The measure $\nu_{\lambda, \varepsilon}$ converges weakly to the measure $\nu_{\lambda}$ as $\varepsilon \to 0$.

Since $\nu_{\lambda}$ is absolutely continuous with respect to Lebesgue measure for almost all $\lambda$ between $\frac{1}{2}$ and 1, but not for all $\lambda$, it is natural to ask how regular $\nu_{\lambda, \varepsilon}$ is. We will prove that $\nu_{\lambda, \varepsilon}$ has density in $L_2$ and that the $L_2$ norm of the density can not grow faster that $1/\sqrt{\varepsilon}$. Let $L$ denote the Lebesgue measure on $\mathbb{R}$. The following theorems will be proved.

**Theorem 2.** For any $\delta > 0$ there exists a constant $c$ such that for any $\varepsilon > 0$ there is a set $E \subset \left(\frac{1}{2} + \varepsilon, 0.6491 - \varepsilon\right)$ with $\mathcal{L}(\left(\frac{1}{2} + \varepsilon, 0.6491 - \varepsilon\right) \setminus E) < \delta$, such that the measure $\nu_{\lambda, \varepsilon}$ satisfies $\|\nu_{\lambda, \varepsilon}\|_2 < c$ for any $\lambda \in E$.

**Theorem 3.** If $\lambda \in \left(\frac{1}{2}, 1\right)$, then the density of $\nu_{\lambda, \varepsilon}$ is in $L_2$ and $\|\nu_{\lambda, \varepsilon}\|_2 < c/\sqrt{\varepsilon}$, for all $0 < \varepsilon < \max\{\lambda - \frac{1}{2}, 1 - \lambda\}$, where $c = \frac{4\varepsilon}{\sqrt{\lambda^2 - \frac{1}{4} - (2\lambda + 1)}}$.

Theorem 2 is proved in Section 2. The proof is only a minor modification of the proof in Peres and Solomyak’s article [4]. The proof of Theorem 3 is in Section 3. It is based on the ideas of transversal intersection of unstable curves in Tsuji’s article [8].

## 2 Proof of Theorem 2

Let $Q = [-1, 1]^3$ and $m \in \mathbb{N}$. We partition the cube $Q$ into the parallelepipeds $\{Q_{0,k}, Q_{1,k}\}_{k=0}^{2^m-1}$, where

$$
Q_{0,k} = \{ (x, y, z) \in Q : y < 0, -1 + k2^{-m+1} \leq z < -1 + (k + 1)2^{-m+1} \},
$$

$$
Q_{1,k} = \{ (x, y, z) \in Q : y \geq 0, -1 + k2^{-m+1} \leq z < -1 + (k + 1)2^{-m+1} \}.
$$

Define $f_{\lambda, \varepsilon, m} : Q \to Q$ by

$$
f_{\lambda, \varepsilon, m} : (x, y, z) \mapsto (\lambda_0(z)x + a(y, z), 2y + b(y), 2^mz + c(z)),
$$

where $\lambda_0(z)$ and $a(y, z)$, $b(y)$, $c(z)$ are defined by

$$
\lambda_0(z) = \lambda + 2^m\varepsilon(z - (-1 + (k + \frac{1}{2})2^{-m+1})),
$$

$$
a(y, z) = \begin{cases} 
-1 + \lambda_0(z) & \text{if } (x, y, z) \in Q_{0,k}, \\
1 - \lambda_0(z) & \text{if } (x, y, z) \in Q_{1,k}.
\end{cases}
$$

$$
b(y) = \begin{cases} 
1 & \text{if } (x, y, z) \in Q_{0,k}, \\
-1 & \text{if } (x, y, z) \in Q_{1,k}.
\end{cases}
$$

$$
c(z) = 2^m - 2k - 1, \quad (x, y, z) \in Q_{0,k} \cup Q_{1,k}.
$$

There is a picture of the map $f_{\lambda, \varepsilon, 2}$ in Figure 1.

For later use we define the sets

$$
Q_0 = \bigcup_{k=0}^{2^m-1} Q_{0,k}, \quad Q_1 = \bigcup_{k=0}^{2^m-1} Q_{1,k}.
$$
Let \( \nu \) be the normalised Lebesgue measure on \( Q \). The measures

\[
\mu_{\lambda, \varepsilon, m, n} = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ f_{\lambda, \varepsilon, m}^{-k}
\]

converges weakly to an srb-measure \( \mu_{\lambda, \varepsilon, m} \) as \( n \to \infty \). The measure \( \mu_{\lambda, \varepsilon, m} \) is clearly ergodic, since if \( p = (x_0, y_0, z_0) \in Q \), then the stable manifold of \( p \) is

\[
\{ (x, y, z) \in Q : y = y_0, z = z_0 \},
\]

and the unstable manifold of \( p \) is

\[
\{ (x, y, z) \in Q : x = x_0 \}.
\]

Moreover, the projection of \( \mu_{\lambda, \varepsilon, m} \) onto the first coordinate is a measure \( \nu_{\lambda, \varepsilon, m} \). More precisely, if \( E \subset [-1, 1] \) is a measurable set, then \( \nu_{\lambda, \varepsilon, m}(E) = \mu_{\lambda, \varepsilon, m}(E \times [-1, 1] \times [-1, 1]) \), defines a measure on \([-1, 1]\).

The measure \( \nu_{\lambda, \varepsilon, m} \) is the distribution of a powerseries \((1 - \lambda - \varepsilon) \sum_{k=0}^{\infty} \partial \lambda_k \)
where \( \lambda_k \) are uniformly distributed in \((\lambda - \varepsilon, \lambda + \varepsilon)\), but not independent. However, the measure \( \nu_{\lambda, \varepsilon, m} \) converges weakly to \( \nu_{\lambda, \varepsilon} \) as \( m \to \infty \).

Let \( A \) be a set of \( 2^{m+1} \) elements. We denote the elements in \( A \) in such a way that \( A = \{ (0, 0), (0, 1), \ldots, (0, 2^m - 1), (1, 0), (1, 1), \ldots, (1, 2^m - 1) \} \). Put \( A_0 = \{ (0, 0), (0, 1), \ldots, (0, 2^m - 1) \} \) and \( A_1 = \{ (1, 0), (1, 1), \ldots, (1, 2^m - 1) \} \).

Let \( \Sigma_0 = A^{\mathbb{N} \cup \{0\}} \). If \( p \) is a point in \( Q \) then there is a unique sequence \( \rho_0(p) = \{ \rho_0(p)_k \}_{k=0}^{\infty} \) in \( \Sigma_0 \) such that

\[
f_{\lambda, \varepsilon, m}(p) \in Q_{\rho_0(p)_k}, \quad k = 0, 1, \ldots.
\]

The map \( \rho_0 : Q \to \Sigma_0 \) is not injective.

We can transfer the measures \( \mu_{\lambda, \varepsilon, m} \) and \( \nu_{\lambda, \varepsilon, m} \) to measures \( \mu_{\Sigma_0} \) and \( \nu_{\Sigma_0} \) on \( \Sigma_0 \) by \( \mu_{\Sigma_0} = \mu_{\lambda, \varepsilon, m} \circ \rho_0^{-1} \) and \( \nu_{\Sigma_0} = \nu_{\lambda, \varepsilon, m} \circ \rho_0^{-1} \).
We let $\Sigma$ denote the natural extension of $\Sigma_0$. That is, $\Sigma$ is the set of all two sided infinite sequences such that any one sided infinite subsequence of a sequence in $\Sigma$ is a sequence in $\Sigma_0$. The measure $\mu_{\Sigma_0}$ defines an ergodic measure $\mu_{\Sigma}$ on $\Sigma$ in a natural way. If $\eta: \Sigma \to \Sigma_0$ is defined by $\eta(\{i_k\}_{k \in \mathbb{Z}}) = \{i_k\}_{k \in \mathbb{N} \cup \{0\}}$, then $\mu_{\Sigma_0}(E) = \mu_{\Sigma}(\eta^{-1}(E))$. Hence, if a function $\varphi$: $\Sigma_0 \to \mathbb{R}$ is $\mu_{\Sigma_0}$-measurable, then the function $\varphi \circ \eta$: $\Sigma \to \mathbb{R}$ is $\mu_\Sigma$-measurable. The map $\varphi_0$: $Q \to \Sigma_0$ can be extended to a map $\varphi$: $Q \to \Sigma$ such that $\varphi(\sigma(a)) = f_{\lambda, e, m}(\varphi(a))$ holds for any sequence $a \in \Sigma$.

Let $B(x, r) = [x - r, x + r]$ be the interval of radius $r$ around $x$. The limit

$$D(x, \lambda, \varepsilon) = \liminf_{r \to 0} \frac{\nu_{\lambda, e, m}(B(x, r))}{2r}$$

is the density of the measure $\nu_{\lambda, e}$ if it exists. If

$$\int_{-1}^{1} D(x, \lambda, \varepsilon) \, d\nu_{\lambda, e}(x) < \infty,$$

then the density $D(x, \lambda, \varepsilon)$ is in $L_2$. We prove that

$$\int_{\frac{1}{2} + \varepsilon + \delta}^{1 - \varepsilon - \delta} \int_{-1}^{1} D(x, \lambda, \varepsilon) \, d\nu_{\lambda, e}(x) \, d\lambda < C,$$

for any $\varepsilon > \delta > 0$ and some constant $C$ which is independent of $m$. This implies that $D(x, \lambda, \varepsilon)$ is in $L_2$ for almost all $\lambda \in (\frac{1}{2} + \varepsilon, 1 - \varepsilon)$, which implies the statement of Theorem 2.

By Fatou's lemma, we get the following inequality.

$$I = \int_{\frac{1}{2} + \varepsilon + \delta}^{1 - \varepsilon - \delta} \int_{-1}^{1} D(x, \lambda, \varepsilon) \, d\nu_{\lambda, e, m}(x) \, d\lambda$$

$$= \int_{\frac{1}{2} + \varepsilon + \delta}^{1 - \varepsilon - \delta} \liminf_{r \to 0} \frac{\nu_{\lambda, e, m}(B(x, r))}{2r} \, d\nu_{\lambda, e}(x) \, d\lambda$$

$$\leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{1 - \varepsilon - \delta} \int_{-1}^{1} \nu_{\lambda, e, m}(B(x, r)) \, d\nu_{\lambda, e, m}(x) \, d\lambda$$

$$= \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{1 - \varepsilon - \delta} \int_{-1}^{1} I_{\{z: |z-x| < r\}}(y) \, d\nu_{\lambda, e, m}(y) \, d\nu_{\lambda, e, m}(x) \, d\lambda,$$

where $I$ denotes the indicator function.

Since $\nu_{\lambda, e, m}$ is the projection of $\mu_{\lambda, e, m}$ to the first coordinate we can rewrite the last line as integrals over the measure $\mu_{\lambda, e, m}$ to get the estimate

$$I \leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{1 - \varepsilon - \delta} \int_{Q} I_{\{z: |z_1-x_1| < r\}}(y) \, d\mu_{\lambda, e, m}(y) \, d\mu_{\lambda, e, m}(x) \, d\lambda,$$

where $x_1$ and $z_1$ denotes the first coordinate of the points $x, z \in Q$. 

We can transference the two innermost integrals to integrals over \(\Sigma\) in order to get the estimate

\[
I \leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta} \int_{\Sigma} I_\epsilon \left[ \left| \varphi(a_1) - \varphi(b_1) \right| < r \right] (b) \ d\mu_{\Sigma}(b) \ d\mu_{\Sigma}(a) \ d\lambda
\]

\[
= \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta} \int_{\Sigma \times \Sigma} I_\epsilon \left[ \left| \varphi(a_1) - \varphi(b_1) \right| < r \right] \ d(\mu_{\Sigma} \times \mu_{\Sigma}) \ d\lambda
\]

The product space \(\Sigma \times \Sigma\) can be written as

\[
\Sigma \times \Sigma = \bigcup_{n=0}^{\infty} \bigcup_{a_{-n} \cdots a_0} \bigcup_{b_{-n} \cdots b_0} [a_{-n} \cdots a_0] \times [b_{-n} \cdots b_0].
\]

Hence

\[
I \leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta} \sum_{n=0}^{\infty} \sum_{a_{-n} \cdots a_0} \sum_{b_{-n} \cdots b_0} H_{a_{-n} \cdots a_0, b_{-n} \cdots b_0} \ d\lambda,
\]

where

\[
H_{a_{-n} \cdots a_0, b_{-n} \cdots b_0} = \int_{[a_{-n} \cdots a_0] \times [b_{-n} \cdots b_0]} I_\epsilon \left[ \left| \varphi(a_1) - \varphi(b_1) \right| < r \right] \ d(\mu_{\Sigma} \times \mu_{\Sigma}).
\]

We can change order of integration to obtain

\[
I \leq \liminf_{r \to 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{a_{-n} \cdots a_0} \sum_{b_{-n} \cdots b_0} J_{a_{-n} \cdots a_0, b_{-n} \cdots b_0},
\]

where

\[
J_{a_{-n} \cdots a_0, b_{-n} \cdots b_0} = \int_{[a_{-n} \cdots a_0] \times [b_{-n} \cdots b_0]} \int_{\frac{1}{2} + \varepsilon + \delta} I_\epsilon \left[ \left| \varphi(a_1) - \varphi(b_1) \right| < r \right] \ d\lambda \ d(\mu_{\Sigma} \times \mu_{\Sigma})(a, b)
\]

If \(a\) and \(b\) are two sequences in \(\Sigma\) with \(\varphi(a) = (x_a, y_a, z_a)\) and \(\varphi(b) = (x_b, y_b, z_b)\), then

\[
x_a = \sum_{k=0}^{\infty} \lambda_k^k (A_{0} - 1)(-1)^{a - k - 1}.
\]
If \( a \) and \( b \) are such that for \(-n < k < 0\) it holds that either \( a_k, b_k \in A_0 \) or \( a_k, b_k \in A_1 \), and \( a_{-n} \in A_0 \) and \( b_{-n} \in A_1 \), then
\[
x_a - x_b = \sum_{k=-n}^{\infty} \lambda_k^0 (\alpha_k - 1) ((-1)^{n+1} - (-1)^{n+k}).
\]

By the methods in Solomyak's paper [7], it follows [6] that this power series has a transversality property, for \( \lambda \in (0.5, 0.6491 - \varepsilon) \). One can use this to estimate that
\[
\int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} I\{ (a, b) : |\rho(a)_1 - \rho(b)_1| < r \} \, d\lambda < C (\frac{1}{2} + \varepsilon + \delta)^{-n},
\]
where \( C \) is a constant. Hence
\[
J_{a_{-n} \ldots a_0, b_{-n} \ldots b_0} \subseteq \int_{[a_{-n} \ldots a_0] \times [b_{-n} \ldots b_0]} C (\frac{1}{2} + \varepsilon + \delta)^{-n} \, d(\mu_x \times \mu_y) = C \frac{4^{n+2} 4^{2n}}{2} (\frac{1}{2} + \varepsilon + \delta)^{-n} \to 0
\]
which implies that
\[
I \leq \liminf_{r \to 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{a_{-n} \ldots a_0 \in A_0 \text{ or } a_{-n} \ldots a_0 \in A_1 \text{, } k > -n} \sum_{b_{-n} \ldots b_0 : a_k \in A_0 \text{ or } a_k \in A_1 \text{ and } b_k \in A_1 \text{ or } b_k \in A_1} C \frac{4^{n+2} 4^{2n}}{2} (\frac{1}{2} + \varepsilon + \delta)^{-n} = C \frac{4^{n+2} 4^{2n}}{2} \sum_{n=0}^{\infty} \left( \frac{1}{2} \frac{1}{2} + \varepsilon + \delta \right)^n < \infty,
\]
which finishes the proof.

3 Proof of Theorem 3

We will use the same notation as in the proof of Theorem 2. We first note that the \( L_2 \) norm of the density of \( \nu_{\lambda, e, m} \) is not larger than twice that of the density of \( \nu_{\lambda, e, m} \). Indeed, if \( \mu_{\lambda, e, m} \) is absolutely continuous with respect to Lebesgue measure, then so is \( \nu_{\lambda, e, m} \). If \( h_{\nu_{\lambda, e, m}}(x) \) and \( h_{\mu_{\lambda, e, m}}(x, y, z) \) denotes the density of \( \nu_{\lambda, e, m} \) and \( \mu_{\lambda, e, m} \) respectively, then by Lyapunov's inequality
\[
\| \nu_{\lambda, e, m} \|_2^2 = \int_{-1}^{1} h_{\nu_{\lambda, e, m}}(x)^2 \, dx = 32 \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} h_{\mu_{\lambda, e, m}}(x, y, z) \frac{dy}{2} \frac{dz}{2} \frac{dx}{2} \leq 32 \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} h_{\mu_{\lambda, e, m}}(x, y, z) \frac{dy}{2} \frac{dz}{2} \frac{dx}{2} = 4 \| \mu_{\lambda, e, m} \|_2^2.
\]
This proves that if $\mu_{\lambda, \epsilon, m}$ has density in $L_2$, then so has $\nu_{\lambda, \epsilon, m}$, and
\[
\|\nu_{\lambda, \epsilon, m}\|_2 \leq 2\|\mu_{\lambda, \epsilon, m}\|_2.
\] (1)
We will therefore bound the $L_2$ norm of $\mu_{\lambda, \epsilon, m}$. For this purpose will need the following transversality property.

**Lemma 3.1.** The cones $C_p = \{ (a, b, c) \in T_p Q : |\frac{a}{c}|, |\frac{b}{c}| < \frac{2^m + 2\epsilon}{2^m - \lambda - \epsilon} \}$, $p \in Q$ defines a family of unstable cones, that is $d_p(\bar{f}_{\lambda, \epsilon, m})(C_p) \subset C_{\bar{f}_{\lambda, \epsilon, m}}(p)$.

Moreover, if $m$ is sufficiently large then the following is true. If $\gamma_1 \subset Q, \gamma_2 \subset Q, \gamma_2$ are two line segments with tangents in $C_p$, such that, $\gamma_1 \notin A_0$ and $\gamma_2 \notin A_1$, then if $f_{\lambda, \epsilon, m}(\gamma_1)$ and $f_{\lambda, \epsilon, m}(\gamma_2)$ intersect, and if $(a_1, b_1, 1)$ and $(a_2, b_2, 1)$ are tangents to $f_{\lambda, \epsilon, m}(\gamma_1)$ and $f_{\lambda, \epsilon, m}(\gamma_2)$ respectively, it holds $|a_1 - a_2| \geq C_{\lambda, \epsilon, m}2^{-(k-1)m}(\lambda - \epsilon)^{k-1} \epsilon$, where
\[
C_{\lambda, \epsilon, m} = \frac{2}{\lambda + \epsilon} - \frac{4(\lambda + \epsilon)}{2^m - \lambda - \epsilon} > 0.
\]

**Proof.** One easily checks that $C_p$ defines a family of unstable cones: The Jacobian of $f_{\lambda, \epsilon, m}$ is
\[
\begin{bmatrix}
\lambda_0 & 0 & 2^{m+1}(x \pm 1)\epsilon \\
0 & 2 & 0 \\
0 & 0 & 2^m
\end{bmatrix}.
\]
If $(a, b, c) \in C_p$, then
\[
d_p(f_{\lambda, \epsilon, m})(a, b, c) = \begin{bmatrix}
\lambda_0 a + 2^{m+1}(x \pm 1)\epsilon c \\
2b \\
2^m c
\end{bmatrix}.
\]
The estimates
\[
\frac{|\lambda_0 a + 2^{m+1}(x \pm 1)\epsilon c|}{2^m c} \leq \frac{\lambda_0 |a|}{2^m} + 4\epsilon \leq \frac{\lambda + \epsilon}{2^m} \frac{2^{m+2} \epsilon}{2^m - \lambda - \epsilon} + 4\epsilon = \frac{2^{m+2} \epsilon}{2^m - \lambda - \epsilon},
\]
\[
\frac{|2b|}{2^m c} \leq \frac{2^{m+2} \epsilon}{2^m - \lambda - \epsilon},
\]
proves that $d_p(f_{\lambda, \epsilon, m})(a, b, c) \subset C_{\bar{f}_{\lambda, \epsilon, m}}(p)$.

Let us now prove the later part of the lemma. Recall that
\[
Q_0 = \bigcup_{k=0}^{2^m-1} Q_{0,k}, \quad Q_1 = \bigcup_{k=0}^{2^m-1} Q_{1,k}.
\]
Take $p = (x, y, z) \in Q$. Then for $(a, b, 1) \in C_p,$
\[
p \in Q_0 \Rightarrow d_q(\bar{f}_{\lambda, \epsilon, m}):(a, b, 1) \mapsto 2^m \left( \frac{\lambda_0 a + (x + 1)\epsilon}{2^m}, \frac{b}{2^m-1}, 1 \right),
\]
\[
p \in Q_1 \Rightarrow d_q(\bar{f}_{\lambda, \epsilon, m}):(a, b, 1) \mapsto 2^m \left( \frac{\lambda_0 a + (x - 1)\epsilon}{2^m}, \frac{b}{2^m-1}, 1 \right).
\]
Assume that \( p = (x_p, y_p, z_p) \in Q_0 \), and \( q = (x_q, y_q, z_q) \in Q_1 \), are such that \( f_{\lambda, \varepsilon, m}(p) = f_{\lambda, \varepsilon, m}(q) = (x, y, z) \). Then \( x_p = \frac{x_p + (1 - \lambda)z_0}{\lambda} \), \( x_q = \frac{x_q + (1 - \lambda)z_0}{\lambda} \), and

\[
\begin{align*}
d_p(f_{\lambda, \varepsilon, m})(C_p) & \subset \{ c(a, b, 1) : \frac{x_p + 1}{\lambda + \varepsilon} - \frac{\lambda + \varepsilon}{2m} - \frac{2m + 2}{\lambda - \varepsilon} < a < \frac{x_p + 1}{\lambda + \varepsilon} + \frac{\lambda + \varepsilon}{2m} - \frac{2m + 2}{\lambda - \varepsilon} \}, \\
d_q(f_{\lambda, \varepsilon, m})(C_q) & \subset \{ c(a, b, 1) : \frac{x_q + 1}{\lambda + \varepsilon} - \frac{\lambda + \varepsilon}{2m} - \frac{2m + 2}{\lambda - \varepsilon} < a < \frac{x_q + 1}{\lambda + \varepsilon} + \frac{\lambda + \varepsilon}{2m} - \frac{2m + 2}{\lambda - \varepsilon} \},
\end{align*}
\]

Hence if \( (a_p, b_p, 1) \in d_p(f_{\lambda, \varepsilon, m})(C_p) \) and \( (a_q, b_q, 1) \in d_q(f_{\lambda, \varepsilon, m})(C_q) \) then

\[
|a_p - a_q| > \frac{2}{\lambda + \varepsilon} - \frac{8\varepsilon(\lambda + \varepsilon)}{2m - \lambda - \varepsilon} = C_{\lambda, \varepsilon, m}\varepsilon,
\]

where

\[
C_{\lambda, \varepsilon, m} = \frac{2}{\lambda + \varepsilon} - \frac{8(\lambda + \varepsilon)}{2m - \lambda - \varepsilon}.
\]

The constant \( C_{\lambda, \varepsilon, m} \) is positive for sufficiently small \( \varepsilon \) provided that

\[
\frac{2}{\lambda} - \frac{8\lambda}{2m - \lambda} > 0 \iff 2m > 4\lambda^2 + \lambda,
\]

which is satisfied if \( m \) is sufficiently large. This proves the lemma in the case \( k = 1 \). The statement for \( k > 1 \) follows by iteration of \( d(f_{\lambda, \varepsilon, m}) \). \[ \square \]

The rest of the proof follows Tsujii’s article [8]. For any \( r > 0 \) we define the bilinear form \( \langle \cdot, \cdot \rangle_r \) of signed measures on \( \mathbb{R} \) by

\[
\langle \varphi_1, \varphi_2 \rangle_r = \int_{\mathbb{R}} \varphi_1(B_r(x))\varphi_2(B_r(x)) \, dx,
\]

where \( B_r(x) = [x - r, x + r] \). One easily proves, see [8], that if

\[
\lim \inf_{r \to \infty} \frac{1}{r^2} \langle \varphi, \varphi \rangle_r < \infty,
\]

then the measure \( \varphi \) has density in \( L_2(\mathbb{R}) \) and

\[
\|\varphi\|_2^2 \leq \lim \inf_{r \to \infty} \frac{1}{r^2} \langle \varphi, \varphi \rangle_r,
\]

where \( \|\varphi\|_2 \) denotes the \( L_2 \) norm of the density of \( \varphi \).

Let \( \mu_\varepsilon \) denote the conditional measure of \( f_{\lambda, \varepsilon, m} \), conditioned on the set \( R_\varepsilon := \{ (a, b, c) \in Q : b = b_0, c = z \} \), where \( b_0 \) is some number such that \( -1 < b_0 < 1 \). Note that \( \mu_\varepsilon \) is independent of \( b_0 \) almost everywhere, so we can forget about \( b_0 \).
Consider the quantity \( f(r) = \frac{1}{r} \int_{-1}^{1} \langle \mu_x, \mu_z \rangle \, dx \). We observe that the measure \( \eta \) on \([-1, 1]\), defined by \( \eta(E) = \mu_{x, \varepsilon, m}([-1, 1] \times [-1, 1] \times E) \) for any Borel set \( E \), is proportional to Lebesgue measure on \([-1, 1]\). This implies that

\[
\|\mu_{x, \varepsilon, m}\|_2^2 = \int_{-1}^{1} \|\mu_z\|_2^2 \, dz \leq \liminf_{r \to 0} f(r).
\]

We will therefore estimate \( f(r) \).

By the invariance of \( \mu_{b, \varepsilon, m} \) it follows that

\[
\mu_z = 2^{-(m+1)} \sum_{a \in A} \mu_{f^{-a}} \circ f^{-a},
\]

where \( f^{-a} \) denotes the inverse of the function \( f_{b, \varepsilon, m} \) restricted to the cylinder \([a]\). By \( f^{-a}(z) \), we mean the unique number \( z_a \) such that there exists numbers \( x_a, y_a, x, y \) with \((x_a, y_a, z_a) \in [a]\) and \( f_{b, \varepsilon, m}(x_a, y_a, z_a) = (x, y, z) \).

The formula (4) allow us to rewrite \( f(r) \) as

\[
f(r) = \frac{1}{r^2} 2^{2(m+1)} \sum_{a, b \in A} \int_{-1}^{1} \langle \mu_{f^{-a}} \circ f^{-a}, \mu_{f^{-b}} \circ f^{-b} \rangle \, dz.
\]

For fixed \( a \) and \( b \) it holds

\[
\langle \mu_{f^{-a}} \circ f^{-a}, \mu_{f^{-b}} \circ f^{-b} \rangle_r \leq \langle \mu_{f^{-a}} \circ f^{-a}, \mu_{f^{-b}} \circ f^{-b} \rangle_f \leq \left( \lambda + \epsilon \right) \langle \mu_{f^{-a}} \circ f^{-a}, \mu_{f^{-b}} \circ f^{-b} \rangle_{\varepsilon, m} \leq \left( \lambda + \epsilon \right) \frac{1}{\pi^2} \left( \lambda + \epsilon \right) \frac{1}{\pi^2} \leq \frac{1}{\pi^2} \frac{1}{\pi^2}.
\]

Moreover

\[
\langle \mu_{f^{-a}} \circ f^{-a}, \mu_{f^{-b}} \circ f^{-b} \rangle_r \leq \int 2rI_{\{0 \leq |\xi| < |\eta| \leq 2r \}}(s, t) \, d\mu_{f^{-a}} \circ f^{-a}(s) \, d\mu_{f^{-b}} \circ f^{-b}(t)
\]

\[
= \int 2rI_{\{0 \leq |\xi| < |\eta| \leq 2r \}}(s, t) \, d\mu_{f^{-a}} \circ f^{-a}(s) \, d\mu_{f^{-b}} \circ f^{-b}(t)
\]
If \( a \) and \( b \) are such that there exists a \( k \) such that either \( a \in A_0 \), \( b \in A_1 \) or \( a \in A_1 \), \( b \in A_0 \), then by Lemma 3.1 and (7) we get that

\[
\int_{-1}^{1} (\mu_{f^{-a}(z)} \circ f^{-a} \cdot \mu_{f^{-b}(z)} \circ f^{-b}) \, dz \\
\leq \int 2r \{ z : |\rho^{-1}(\cdots c_{\cdot 2} a \varphi_0(z)) - \rho^{-1}(\cdots d_{2} b \varphi_0(z))| < 2r \} \\
\leq 8r^2 \frac{1}{C_{\lambda, \varepsilon, m}},
\]

and so since there are at most \( 2^{2(m+1)} \) pairs \( a, b \) such that either \( a \in A_0 \), \( b \in A_1 \) or \( a \in A_1 \), \( b \in A_0 \), we get that

\[
\sum_{a \in A_0, b \in A_1 \text{ or } a \in A_1, b \in A_0} \int_{-1}^{1} (\mu_{f^{-a}(z)} \circ f^{-a} \cdot \mu_{f^{-b}(z)} \circ f^{-b}) \, dz \leq \frac{8r^2}{C_{\lambda, \varepsilon, m}} 2^{2(m+1)}.
\]

For the case that \( a, b \in A_0 \) or \( a, b \in A_1 \) we use (6) to estimate that

\[
\sum_{a, b \in A_0 \text{ or } a, b \in A_1} (\mu_{f^{-a}(z)} \circ f^{-a} \cdot \mu_{f^{-b}(z)} \circ f^{-b}) \leq (\lambda + \varepsilon) 2^m \sum_{a} (\mu_{f^{-a}(z)} \circ f^{-a}) \frac{r}{\lambda - \varepsilon}.
\]

Now, (5) and the last two estimates implies

\[
f(r) \leq \frac{8}{C_{\lambda, \varepsilon, m}} + 2^{-2(m+1)} (\lambda + \varepsilon) 2^m \frac{1}{r^2} \sum_{a} \int_{-1}^{1} (\mu_{f^{-a}(z)} \circ f^{-a}) \frac{r}{\lambda - \varepsilon} \, dz \\
= \frac{8}{C_{\lambda, \varepsilon, m}} + 2^{-2(m+1)} \frac{\lambda + \varepsilon}{(\lambda - \varepsilon)^2} 2^m \frac{1}{(r^2)^2} \sum_{a} 2^m \int_{a} (\mu_{a} \circ \mu_{a}) \frac{r}{\lambda - \varepsilon} \, dz \\
= \frac{8r^2}{C_{\lambda, \varepsilon, m}} + 2^{-2}(\lambda - \varepsilon)^{-2} \left( \frac{\lambda + \varepsilon}{\lambda - \varepsilon} \right) f \left( \frac{r}{\lambda - \varepsilon} \right).
\]

Hence

\[
\frac{f(r)}{r^2} \leq \frac{8}{C_{\lambda, \varepsilon, m}} \sum_{n=0}^{\infty} (2(\lambda - \varepsilon))^{-2n} \left( \frac{\lambda + \varepsilon}{\lambda - \varepsilon} \right)^n \\
= \frac{8}{C_{\lambda, \varepsilon, m}} (\lambda - \varepsilon)^2 - \frac{1}{4} \frac{\lambda + \varepsilon}{\lambda - \varepsilon}.
\]

By (3) we thus have

\[
\|\mu_{\varepsilon, m}\|_2 \leq \frac{\varepsilon_m}{\sqrt{\varepsilon}}
\]
with
\[ c_m = \sqrt{\frac{8}{(\lambda - \epsilon)^2} - \frac{\lambda + \epsilon}{\lambda - \epsilon} \left( \frac{2}{\lambda + \epsilon} - \frac{4(\lambda + \epsilon)}{2m - \lambda - \epsilon} \right)^{-1}}. \]

By (1) we have
\[ \| \mathcal{Y}_{\lambda, \epsilon, m} \|_2 \leq 2 \| \mathcal{Y}_{\lambda, \epsilon, m} \|_2 \leq \frac{c_m}{\sqrt{\epsilon}}. \]

Letting \( m \to \infty \) we find that
\[ \| \mathcal{Y}_{\lambda, \epsilon} \|_2 \leq \frac{c}{\sqrt{\epsilon}}. \]

with
\[ c = 4 \sqrt{\frac{(\lambda^2 - \epsilon^2)(\lambda - \epsilon)}{(\lambda - \epsilon)^2 - \frac{\lambda + \epsilon}{4} \lambda - \epsilon}} \leq \frac{4 \lambda \sqrt{\lambda}}{\sqrt{\lambda^2 - \frac{1}{4} - (2\lambda + 1) \epsilon}}. \]
References


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