On Random Bernoulli Convolutions

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With one figure

Abstract

We study the distribution of the random series $\sum_{k=0}^{\infty} \pm \lambda_k^k$ where λ_k are independently and uniformly distributed in $(\lambda - \varepsilon, \lambda + \varepsilon)$. It is proved that the distribution of the series has density in L_2 and that the L_2 norm of the density does not grow faster than $1/\sqrt{\varepsilon}$.

1 Introduction and Statements of Results

Let $\lambda \in (\frac{1}{2}, 1)$. The distribution of the random variable Y_λ , defined by

$$Y_\lambda = (1 - \lambda) \sum_{k=0}^{\infty} \vartheta_k \lambda^k,$$

where the random variables ϑ_k are independent and identically distributed according to $P(\vartheta_k = +1) = P(\vartheta_k = -1) = 1/2$, is denoted by ν_λ . We have chosen the constant $(1 - \lambda)$ in front of the sum in order to have ν_λ with the support $[-1, 1]$ for all λ . This distribution was studied by Erdős in [1] and [2]. It is known that ν_λ is absolutely continuous with respect to Lebesgue measure for almost all $\lambda \in (\frac{1}{2}, 1)$, see [7] and [4], and that if λ^{-1} is a Pisot number, then ν_λ is singular with respect to Lebesgue measure, [1]. It is also known that for $\lambda = 1/\sqrt[k]{2}$, $k \in \mathbb{N}$, and for some algebraic integers, ν_λ is absolutely continuous with respect to Lebesgue measure, [3], [9]. There is a nice survey of the subject in [5].

For $\varepsilon > 0$, we study the distribution of the random variable

$$Y_{\lambda, \varepsilon} = (1 - \lambda - \varepsilon) \sum_{k=0}^{\infty} \vartheta_k \lambda_k^k,$$

where the random variables λ_k are uniformly and independently distributed in $(\lambda - \varepsilon, \lambda + \varepsilon)$, and the random variables ϑ_k are independent, independent of λ_k and distributed as above. We let $\nu_{\lambda, \varepsilon}$ denote the distribution of the random variable $Y_{\lambda, \varepsilon}$. The coefficient $1 - \lambda - \varepsilon$ in the definition of $Y_{\lambda, \varepsilon}$ is chosen in order that the measure $\nu_{\lambda, \varepsilon}$ has support in $[-1, 1]$.

It is not hard to see that the distribution $\nu_{\lambda,\varepsilon}$ approximates ν_λ . More precisely, one easily proves the following theorem.

Theorem 1. The measure $\nu_{\lambda,\varepsilon}$ converges weakly to the measure ν_λ as $\varepsilon \rightarrow 0$.

Since ν_λ is absolutely continuous with respect to Lebesgue measure for almost all λ between $\frac{1}{2}$ and 1, but not for all λ , it is natural to ask how regular $\nu_{\lambda,\varepsilon}$ is. We will prove that $\nu_{\lambda,\varepsilon}$ has density in L_2 and that the L_2 norm of the density can not grow faster than $1/\sqrt{\varepsilon}$. Let $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}$. The following theorems will be proved.

Theorem 2. For any $\delta > 0$ there exists a constant c such that for any $\varepsilon > 0$ there is a set $E \subset (\frac{1}{2} + \varepsilon, 0.6491 - \varepsilon)$ with $\mathcal{L}((\frac{1}{2} + \varepsilon, 0.6491 - \varepsilon) \setminus E) < \delta$, such that the measure $\nu_{\lambda,\varepsilon}$ satisfies $\|\nu_{\lambda,\varepsilon}\|_2 < c$ for any $\lambda \in E$.

Theorem 3. If $\lambda \in (\frac{1}{2}, 1)$, then the density of $\nu_{\lambda,\varepsilon}$ is in L_2 and $\|\nu_{\lambda,\varepsilon}\|_2 < c/\sqrt{\varepsilon}$, for all $0 < \varepsilon < \max\{\lambda - \frac{1}{2}, 1 - \lambda\}$, where $c = \frac{4\lambda\sqrt{\lambda}}{\sqrt{\lambda^2 - \frac{1}{4} - (2\lambda+1)\varepsilon}}$.

Theorem 2 is proved in Section 2. The proof is only a minor modification of the proof in Peres' and Solomyak's article [4]. The proof of Theorem 3 is in Section 3. It is based on the ideas of transversal intersection of unstable curves in Tsujii's article [8].

2 Proof of Theorem 2

Let $Q = [-1, 1)^3$ and $m \in \mathbb{N}$. We partition the cube Q into the parallelepipeds $\{Q_{0,k}, Q_{1,k}\}_{k=0}^{2^m-1}$, where

$$\begin{aligned} Q_{0,k} &= \{(x, y, z) \in Q : y < 0, -1 + k2^{-m+1} \leq z < -1 + (k+1)2^{-m+1}\}, \\ Q_{1,k} &= \{(x, y, z) \in Q : y \geq 0, -1 + k2^{-m+1} \leq z < -1 + (k+1)2^{-m+1}\}. \end{aligned}$$

Define $f_{\lambda,\varepsilon,m}: Q \rightarrow Q$ by

$$f_{\lambda,\varepsilon,m}: (x, y, z) \mapsto (\lambda_0(z)x + a(y, z), 2y + b(y), 2^m z + c(z)),$$

where $\lambda_0(z)$ and $a(y, z)$, $b(y)$, $c(z)$ are defined by

$$\begin{aligned} \lambda_0(z) &= \lambda + 2^{m+1}\varepsilon(z - (-1 + (k + \frac{1}{2})2^{-m+1})), \quad (x, y, z) \in Q_{0,k} \cup Q_{1,k} \\ a(y, z) &= \begin{cases} -1 + \lambda_0(z) & \text{if } (x, y, z) \in Q_{0,k} \\ 1 - \lambda_0(z) & \text{if } (x, y, z) \in Q_{1,k} \end{cases}, \\ b(y) &= \begin{cases} 1 & \text{if } (x, y, z) \in Q_{0,k} \\ -1 & \text{if } (x, y, z) \in Q_{1,k} \end{cases}, \\ c(z) &= 2^m - 2k - 1, \quad (x, y, z) \in Q_{0,k} \cup Q_{1,k}. \end{aligned}$$

There is a picture of the map $f_{\lambda,\varepsilon,2}$ in Figure 1.

For later use we define the sets

$$Q_0 = \bigcup_{k=0}^{2^m-1} Q_{0,k}, \quad Q_1 = \bigcup_{k=0}^{2^m-1} Q_{1,k}.$$

Figure 1: The action of the map $f_{\lambda, \varepsilon, 2}$ restricted to the sets $Q_{0,0}$, $Q_{0,1}$, $Q_{1,0}$ and $Q_{1,1}$, for $\lambda = 3/4$ and $\varepsilon = 1/10$.

Let ν be the normalised Lebesgue measure on Q . The measures

$$\mu_{\lambda, \varepsilon, m, n} = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ f_{\lambda, \varepsilon, m}^{-k},$$

converges weakly to an SRB-measure $\mu_{\lambda, \varepsilon, m}$ as $n \rightarrow \infty$. The measure $\mu_{\lambda, \varepsilon, m}$ is clearly ergodic, since if $p = (x_0, y_0, z_0) \in Q$, then the stable manifold of p is

$$\{ (x, y, z) \in Q : y = y_0, z = z_0 \},$$

and the unstable manifold of p is

$$\{ (x, y, z) \in Q : x = x_0 \}.$$

Moreover, the projection of $\mu_{\lambda, \varepsilon, m}$ onto the first coordinate is a measure $\nu_{\lambda, \varepsilon, m}$. More precisely, if $E \subset [-1, 1)$ is a measurable set, then $\nu_{\lambda, \varepsilon, m}(E) = \mu_{\lambda, \varepsilon, m}(E \times [-1, 1) \times [-1, 1))$, defines a measure on $[-1, 1)$.

The measure $\nu_{\lambda, \varepsilon, m}$ is the distribution of a powerseries $(1 - \lambda - \varepsilon) \sum_{k=0}^{\infty} \vartheta_k \lambda_k^k$ where λ_k are uniformly distributed in $(\lambda - \varepsilon, \lambda + \varepsilon)$, but not independent. However, the measure $\nu_{\lambda, \varepsilon, m}$ converges weakly to $\nu_{\lambda, \varepsilon}$ as $m \rightarrow \infty$.

Let A be a set of 2^{m+1} elements. We denote the elements in A in such a way that $A = \{ (0, 0), (0, 1), \dots, (0, 2^m - 1), (1, 0), (1, 1), \dots, (1, 2^m - 1) \}$. Put $A_0 = \{ (0, 0), (0, 1), \dots, (0, 2^m - 1) \}$ and $A_1 = \{ (1, 0), (1, 1), \dots, (1, 2^m - 1) \}$.

Let $\Sigma_0 = A^{\mathbb{N} \cup \{0\}}$. If p is a point in Q then there is a unique sequence $\rho_0(p) = \{\rho_0(p)_k\}_{k=0}^{\infty} \in \Sigma_0$ such that

$$f_{\lambda, \varepsilon, m}(p) \in Q_{\rho_0(p)_k}, \quad k = 0, 1, \dots$$

The map $\rho_0: Q \rightarrow \Sigma_0$ is not injective.

We can transfe the measures $\mu_{\lambda, \varepsilon, m}$ and $\nu_{\lambda, \varepsilon, m}$ to measures μ_{Σ_0} and ν_{Σ_0} on Σ_0 by $\mu_{\Sigma_0} = \mu_{\lambda, \varepsilon, m} \circ \rho_0^{-1}$ and $\nu_{\Sigma_0} = \nu_{\lambda, \varepsilon, m} \circ \rho_0^{-1}$.

We let Σ denote the natural extension of Σ_0 . That is, Σ is the set of all two sided infinite sequences such that any one sided infinite subsequence of a sequence in Σ is a sequence in Σ_0 . The measure μ_{Σ_0} defines an ergodic measure μ_Σ on Σ in a natural way. If $\eta: \Sigma \rightarrow \Sigma_0$ is defined by $\eta(\{i_k\}_{k \in \mathbb{Z}}) = \{i_k\}_{k \in \mathbb{N} \cup \{0\}}$, then $\mu_{\Sigma_0}(E) = \mu_\Sigma(\eta^{-1}(E))$. Hence, if a function $\varphi: \Sigma_0 \rightarrow \mathbb{R}$ is μ_{Σ_0} -measurable, then the function $\varphi \circ \eta: \Sigma \rightarrow \mathbb{R}$ is μ_Σ -measurable. The map $\rho_0: Q \rightarrow \Sigma_0$ can be extended to a map $\rho: Q \rightarrow \Sigma$ such that $\rho(\sigma(a)) = f_{\lambda, \varepsilon, m}(\rho(a))$ holds for any sequence $a \in \Sigma$.

Let $B(x, r) = [x - r, x + r]$ be the interval of radius r around x . The limit

$$D(x, \lambda, \varepsilon) = \liminf_{r \rightarrow 0} \frac{\nu_{\lambda, \varepsilon, m}(B(x, r))}{2r}$$

is the density of the measure $\nu_{\lambda, \varepsilon}$ if it exists. If

$$\int_{-1}^1 D(x, \lambda, \varepsilon) d\nu_{\lambda, \varepsilon, m}(x) < \infty,$$

then the density $D(x, \lambda, \varepsilon)$ is in L_2 . We prove that

$$\int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_{-1}^1 D(x, \lambda, \varepsilon) d\nu_{\lambda, \varepsilon, m}(x) d\lambda < C,$$

for any $\varepsilon > \delta > 0$ and some constant C which is independent of m . This implies that $D(x, \lambda, \varepsilon)$ is in L_2 for almost all $\lambda \in (\frac{1}{2} + \varepsilon, 1 - \varepsilon)$, which implies the statement of Theorem 2.

By Fatou's lemma, we get the following inequality.

$$\begin{aligned} I &= \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_{-1}^1 D(x, \lambda, \varepsilon) d\nu_{\lambda, \varepsilon, m}(x) d\lambda \\ &= \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_{-1}^1 \liminf_{r \rightarrow 0} \frac{\nu_{\lambda, \varepsilon, m}(B(x, r))}{2r} d\nu_{\lambda, \varepsilon}(x) d\lambda \\ &\leq \liminf_{r \rightarrow 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_{-1}^1 \nu_{\lambda, \varepsilon, m}(B(x, r)) d\nu_{\lambda, \varepsilon, m}(x) d\lambda \\ &= \liminf_{r \rightarrow 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_{-1}^1 \int_{-1}^1 I_{\{z: |x-z| < r\}}(y) d\nu_{\lambda, \varepsilon, m}(y) d\nu_{\lambda, \varepsilon, m}(x) d\lambda, \end{aligned}$$

where I denotes the indicator function.

Since $\nu_{\lambda, \varepsilon, m}$ is the projection of $\mu_{\lambda, \varepsilon, m}$ to the first coordinate we can rewrite the last line as integrals over the measure $\mu_{\lambda, \varepsilon, m}$ to get the estimate

$$I \leq \liminf_{r \rightarrow 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_Q \int_Q I_{\{z: |x_1 - z_1| < r\}}(y) d\mu_{\lambda, \varepsilon, m}(y) d\mu_{\lambda, \varepsilon, m}(x) d\lambda,$$

where x_1 and z_1 denotes the first coordinate of the points $x, z \in Q$.

We can transference the two innermost integrals to integrals over Σ in order to get the estimate

$$\begin{aligned} I &\leq \liminf_{r \rightarrow 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_{\Sigma} \int_{\Sigma} I_{\{c: |\rho(a)_1 - \rho(c)_1| < r\}}(b) d\mu_{\Sigma}(b) d\mu_{\Sigma}(a) d\lambda \\ &= \liminf_{r \rightarrow 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \int_{\Sigma \times \Sigma} I_{\{(a,b): |\rho(a)_1 - \rho(b)_1| < r\}} d(\mu_{\Sigma} \times \mu_{\Sigma}) d\lambda \end{aligned}$$

The product space $\Sigma \times \Sigma$ can be written as

$$\Sigma \times \Sigma = \bigcup_{n=0}^{\infty} \bigcup_{a_{-n} \cdots a_0} \bigcup_{\substack{b_{-n} \cdots b_0 \\ a_k, b_k \in A_0 \text{ or } a_k, b_k \in A_1, k > -n \\ \{a_{-n}, b_{-n}\} \not\subset A_0 \text{ and } \{a_{-n}, b_{-n}\} \not\subset A_1}} [a_{-n} \cdots a_0] \times [b_{-n} \cdots b_0].$$

Hence

$$I \leq \liminf_{r \rightarrow 0} \frac{1}{2r} \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} \sum_{n=0}^{\infty} \sum_{a_{-n} \cdots a_0} \sum_{\substack{b_{-n} \cdots b_0 \\ a_k, b_k \in A_0 \text{ or } a_k, b_k \in A_1, k > -n \\ \{a_{-n}, b_{-n}\} \not\subset A_0 \text{ and } \{a_{-n}, b_{-n}\} \not\subset A_1}} H_{a_{-n} \cdots a_0, b_{-n} \cdots b_0} d\lambda,$$

where

$$H_{a_{-n} \cdots a_0, b_{-n} \cdots b_0} = \int_{[a_{-n} \cdots a_0] \times [b_{-n} \cdots b_0]} I_{\{(a,b): |\rho(a)_1 - \rho(b)_1| < r\}} d(\mu_{\Sigma} \times \mu_{\Sigma}).$$

We can change order of integration to obtain

$$I \leq \liminf_{r \rightarrow 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{a_{-n} \cdots a_0} \sum_{\substack{b_{-n} \cdots b_0 \\ a_k, b_k \in A_0 \text{ or } a_k, b_k \in A_1, k > -n \\ \{a_{-n}, b_{-n}\} \not\subset A_0 \text{ and } \{a_{-n}, b_{-n}\} \not\subset A_1}} J_{a_{-n} \cdots a_0, b_{-n} \cdots b_0},$$

where

$$\begin{aligned} J_{a_{-n} \cdots a_0, b_{-n} \cdots b_0} &= \int_{[a_{-n} \cdots a_0] \times [b_{-n} \cdots b_0]} \int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} I_{\{(a,b): |\rho(a)_1 - \rho(b)_1| < r\}} d\lambda d(\mu_{\Sigma} \times \mu_{\Sigma})(a, b) \end{aligned}$$

If a and b are two sequences in Σ with $\rho(a) = (x_a, y_a, z_a)$ and $\rho(b) = (x_b, y_b, z_b)$, then

$$x_a = \sum_{k=0}^{\infty} \lambda_0^k (\lambda_0 - 1) (-1)^{a-k-1}.$$

If a and b are such that for $-n < k < 0$ it holds that either $a_k, b_k \in A_0$ or $a_k, b_k \in A_1$, and $a_{-n} \in A_0$ and $b_{-n} \in A_1$, then

$$x_a - x_b = \sum_{k=n}^{\infty} \lambda_0^k (\lambda_0 - 1) ((-1)^{a-k} - (-1)^{b-k}).$$

By the methods in Solomyak's paper [7], it follows [6] that this power series has a transversality property, for $\lambda \in (0.5, 0.6491 - \varepsilon)$. One can use this to estimate that

$$\int_{\frac{1}{2} + \varepsilon + \delta}^{0.6491 - \varepsilon - \delta} I_{\{(a,b) : |\rho(a)_1 - \rho(b)_1| < r\}} d\lambda < Cr \left(\frac{1}{2} + \varepsilon + \delta\right)^{-n},$$

where C is a constant. Hence

$$\begin{aligned} J_{a_{-n} \cdots a_0, b_{-n} \cdots b_0} &\leq \int_{[a_{-n} \cdots a_0] \times [b_{-n} \cdots b_0]} Cr \left(\frac{1}{2} + \delta - \varepsilon\right)^{-n} d(\mu_{\Sigma} \times \mu_{\Sigma}) \\ &= Cr 4^{-2n} \left(\frac{1}{2} + \varepsilon + \delta\right)^{-n}. \end{aligned}$$

This implies that

$$\begin{aligned} I &\leq \liminf_{r \rightarrow 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{a_{-n} \cdots a_0} \sum_{\substack{b_{-n} \cdots b_0 \\ a_k, b_k \in A_0 \text{ or } a_k, b_k \in A_1, k > -n \\ \{a_{-n}, b_{-n}\} \not\subset A_0 \text{ and } \{a_{-n}, b_{-n}\} \not\subset A_1}} Cr 4^{-2n} \left(\frac{1}{2} + \varepsilon + \delta\right)^{-n} \\ &= \frac{C}{2} \sum_{n=0}^{\infty} 4^n 2^n C 4^{-2n} \left(\frac{1}{2} + \delta - \varepsilon\right)^{-n} = \frac{C}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2(\frac{1}{2} + \varepsilon + \delta)}\right)^n < \infty, \end{aligned}$$

which finishes the proof.

3 Proof of Theorem 3

We will use the same notation as in the proof of Theorem 2. We first note that the L_2 norm of the density of $\nu_{\lambda, \varepsilon, m}$ is not larger than twice that of the density of $\mu_{\lambda, \varepsilon, m}$. Indeed, if $\mu_{\lambda, \varepsilon, m}$ is absolutely continuous with respect to Lebesgue measure, then so is $\nu_{\lambda, \varepsilon, m}$. If $h_{\nu_{\lambda, \varepsilon, m}}(x)$ and $h_{\mu_{\lambda, \varepsilon, m}}(x, y, z)$ denotes the density of $\nu_{\lambda, \varepsilon, m}$ and $\mu_{\lambda, \varepsilon, m}$ respectively, then by Lyapunov's inequality

$$\begin{aligned} \|\nu_{\lambda, \varepsilon, m}\|_2^2 &= \int_{-1}^1 h_{\nu_{\lambda, \varepsilon, m}}(x)^2 dx \\ &= 32 \int_{-1}^1 \left(\int_{-1}^1 \int_{-1}^1 h_{\mu_{\lambda, \varepsilon, m}}(x, y, z) \frac{dy}{2} \frac{dz}{2} \right)^2 \frac{dx}{2} \\ &\leq 32 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 h_{\mu_{\lambda, \varepsilon, m}}(x, y, z)^2 \frac{dy}{2} \frac{dz}{2} \frac{dx}{2} = 4 \|\mu_{\lambda, \varepsilon, m}\|_2^2. \end{aligned}$$

This proves that if $\mu_{\lambda,\varepsilon,m}$ has density in L_2 , then so has $\nu_{\lambda,\varepsilon,m}$, and

$$\|\nu_{\lambda,\varepsilon,m}\|_2 \leq 2\|\mu_{\lambda,\varepsilon,m}\|_2. \quad (1)$$

We will therefore bound the L_2 norm of $\mu_{\lambda,\varepsilon,m}$. For this purpose will need the following transversality property.

Lemma 3.1. The cones $C_p = \{(a, b, c) \in T_p Q : |\frac{a}{c}|, |\frac{b}{c}| < \frac{2^{m+2}\varepsilon}{2^m - \lambda - \varepsilon}\}$, $p \in Q$ defines a family of unstable cones, that is $d_p(f_{\lambda,\varepsilon,m})(C_p) \subset C_{f_{\lambda,\varepsilon,m}(p)}$.

Moreover, if m is sufficiently large then the following is true. If $\gamma_1 \subset Q_{\gamma_1}$ and $\gamma_2 \subset Q_{\gamma_2}$ are two line segments with tangents in C_p , such that, $\gamma_1 \in A_0$ and $\gamma_2 \in A_1$, then if $f_{\lambda,\varepsilon,m}(\gamma_1)$ and $f_{\lambda,\varepsilon,m}(\gamma_2)$ intersect, and if $(a_1, b_1, 1)$ and $(a_2, b_2, 1)$ are tangents to $f_{\lambda,\varepsilon,m}^k(\gamma_1)$ and $f_{\lambda,\varepsilon,m}^k(\gamma_2)$ respectively, it holds $|a_1 - a_2| > C_{\lambda,\varepsilon,m} 2^{-(k-1)m} (\lambda - \varepsilon)^{k-1} \varepsilon$, where

$$C_{\lambda,\varepsilon,m} = \frac{2}{\lambda + \varepsilon} - \frac{4(\lambda + \varepsilon)}{2^m - \lambda - \varepsilon} > 0.$$

Proof. One easily checks that C_p defines a family of unstable cones: The Jacobian of $f_{\lambda,\varepsilon,m}$ is

$$\begin{bmatrix} \lambda_0 & 0 & 2^{m+1}(x \pm 1)\varepsilon \\ 0 & 2 & 0 \\ 0 & 0 & 2^m \end{bmatrix}.$$

If $(a, b, c) \in C_p$, then

$$d_p(f_{\lambda,\varepsilon,m})(a, b, c) = \begin{bmatrix} \lambda_0 a + 2^{m+1}(x \pm 1)\varepsilon c \\ 2b \\ 2^m c \end{bmatrix}.$$

The estimates

$$\begin{aligned} \frac{|\lambda_0 a + 2^{m+1}(x \pm 1)\varepsilon c|}{|2^m c|} &\leq \frac{\lambda_0}{2^m} \frac{|a|}{|c|} + 4\varepsilon \leq \frac{\lambda + \varepsilon}{2^m} \frac{2^{m+2}\varepsilon}{2^m - \lambda - \varepsilon} + 4\varepsilon \\ &= \frac{2^{m+2}\varepsilon}{2^m - \lambda - \varepsilon}, \\ \frac{|2b|}{|2^m c|} &< \frac{2^{m+2}\varepsilon}{2^m - \lambda - \varepsilon}, \end{aligned}$$

proves that $d_p(f_{\lambda,\varepsilon,m})(a, b, c) \subset C_{f_{\lambda,\varepsilon,m}(p)}$.

Let us now prove the later part of the lemma. Recall that

$$Q_0 = \bigcup_{k=0}^{2^m-1} Q_{0,k}, \quad Q_1 = \bigcup_{k=0}^{2^m-1} Q_{1,k}.$$

Take $p = (x, y, z) \in Q$. Then for $(a, b, 1) \in C_p$.

$$\begin{aligned} p \in Q_0 &\Rightarrow d_p f_{\lambda,\varepsilon,m} : (a, b, 1) \mapsto 2^m \left(\frac{\lambda_0}{2^m} a + (x+1)\varepsilon, \frac{b}{2^{m-1}}, 1 \right), \\ p \in Q_1 &\Rightarrow d_p f_{\lambda,\varepsilon,m} : (a, b, 1) \mapsto 2^m \left(\frac{\lambda_0}{2^m} a + (x-1)\varepsilon, \frac{b}{2^{m-1}}, 1 \right). \end{aligned}$$

Assume that $p = (x_p, y_p, z_p) \in Q_0$, and $q = (x_q, y_q, z_q) \in Q_1$, are such that $f_{\lambda, \varepsilon, m}(p) = f_{\lambda, \varepsilon, m}(q) = (x, y, z)$.

Then $x_p = \frac{x+(1-\lambda_0)}{\lambda_0}$, $x_q = \frac{x-(1-\lambda_0)}{\lambda_0}$ and

$$\begin{aligned} & d_p(f_{\lambda, \varepsilon, m})(C_p) \\ & \subset \left\{ c(a, b, 1) : \frac{x+1}{\lambda+\varepsilon} \varepsilon - \frac{\lambda+\varepsilon}{2^m} \frac{2^{m+2} \varepsilon}{2^m - \lambda - \varepsilon} < a < \frac{x+1}{\lambda-\varepsilon} \varepsilon + \frac{\lambda+\varepsilon}{2^m} \frac{2^{m+2} \varepsilon}{2^m - \lambda - \varepsilon} \right\}, \\ & d_q(f_{\lambda, \varepsilon, m})(C_q) \\ & \subset \left\{ c(a, b, 1) : \frac{x-1}{\lambda-\varepsilon} \varepsilon - \frac{\lambda+\varepsilon}{2^m} \frac{2^{m+2} \varepsilon}{2^m - \lambda - \varepsilon} < a < \frac{x-1}{\lambda+\varepsilon} \varepsilon + \frac{\lambda+\varepsilon}{2^m} \frac{2^{m+2} \varepsilon}{2^m - \lambda - \varepsilon} \right\}. \end{aligned}$$

Hence if $(a_p, b_p, 1) \in d_p(f_{\lambda, \varepsilon, m})(C_p)$ and $(a_q, b_q, 1) \in d_q(f_{\lambda, \varepsilon, m})(C_q)$ then

$$|a_p - a_q| > \frac{2}{\lambda + \varepsilon} \varepsilon - \frac{8\varepsilon(\lambda + \varepsilon)}{2^m - \lambda - \varepsilon} = C_{\lambda, \varepsilon, m} \varepsilon, \quad (2)$$

where

$$C_{\lambda, \varepsilon, m} = \frac{2}{\lambda + \varepsilon} - \frac{8(\lambda + \varepsilon)}{2^m - \lambda - \varepsilon}.$$

The constant $C_{\lambda, \varepsilon, m}$ is positive for sufficiently small ε provided that

$$\frac{2}{\lambda} - \frac{8\lambda}{2^m - \lambda} > 0 \quad \Leftrightarrow \quad 2^m > 4\lambda^2 + \lambda,$$

which is satisfied if m is sufficiently large. This proves the lemma in the case $k = 1$. The statement for $k > 1$ follows by iteration of $d(f_{\lambda, \varepsilon, m})$. $\square$

The rest of the proof follows Tsujii's article [8]. For any $r > 0$ we define the bilinear form $(\cdot, \cdot)_r$ of signed measures on $\mathbb{R}$ by

$$(\rho_1, \rho_2)_r = \int_{\mathbb{R}} \rho_1(B_r(x)) \rho_2(B_r(x)) dx,$$

where $B_r(x) = [x - r, x + r]$. One easily proves, see [?], that if

$$\liminf_{r \rightarrow \infty} \frac{1}{r^2} (\rho, \rho)_r < \infty,$$

then the measure ρ has density in $L_2(\mathbb{R})$ and

$$\|\rho\|_2^2 \leq \liminf_{r \rightarrow \infty} \frac{1}{r^2} (\rho, \rho)_r,$$

where $\|\rho\|_2$ denotes the L_2 norm of the density of ρ .

Let μ_z denote the conditional measure of $\mu_{\lambda, \varepsilon, m}$, conditioned on the set $R_z := \{(a, b, c) \in Q : b = b_0, c = z\}$, where b_0 is some number such that $-1 < b_0 < 1$. Note that μ_z is independent of b_0 almost everywhere, so we can forget about b_0 .

Consider the quantity $J(r) = \frac{1}{r^2} \int_{-1}^1 (\mu_z, \mu_z)_r dz$. We observe that the measure η on $[-1, 1]$, defined by $\eta(E) = \mu_{\lambda, \varepsilon, m}([-1, 1] \times [-1, 1] \times E)$ for any Borel set E , is proportional to Lebesgue measure on $[-1, 1]$. This implies that

$$\|\mu_{\lambda, \varepsilon, m}\|_2^2 = \int_{-1}^1 \|\mu_z\|_2^2 dz \leq \liminf_{r \rightarrow 0} J(r). \quad (3)$$

We will therefore estimate $J(r)$.

By the invariance of $\mu_{\lambda, \varepsilon, m}$ it follows that

$$\mu_z = 2^{-(m+1)} \sum_{a \in A} \mu_{f^{-a}(z)} \circ f^{-a}, \quad (4)$$

where f^{-a} denotes the inverse of the function $f_{\lambda, \varepsilon, m}$ restricted to the cylinder $[a]$. By $f^{-a}(z)$, we mean the unique number z_a such that there exists numbers x_a, y_a, x, y with $(x_a, y_a, z_a) \in [a]$ and $f_{\lambda, \varepsilon, m}(x_a, y_a, z_a) = (x, y, z)$.

The formula (4) allow us to rewrite $J(r)$ as

$$J(r) = \frac{1}{r^2} 2^{-2(m+1)} \sum_{a, b \in A} \int_{-1}^1 (\mu_{f^{-a}(z)} \circ f^{-a}, \mu_{f^{-b}(z)} \circ f^{-b})_r dz. \quad (5)$$

For fixed a and b it holds

$$\begin{aligned} & (\mu_{f^{-a}(z)} \circ f^{-a}, \mu_{f^{-b}(z)} \circ f^{-b})_r \\ & \leq (\mu_{f^{-a}(z)} \circ f^{-a}, \mu_{f^{-a}(z)} \circ f^{-a})_r^{\frac{1}{2}} (\mu_{f^{-b}(z)} \circ f^{-b}, \mu_{f^{-b}(z)} \circ f^{-b})_r^{\frac{1}{2}} \\ & \leq ((\lambda + \varepsilon)(\mu_{f^{-a}(z)}, \mu_{f^{-a}(z)})_{\frac{r}{\lambda - \varepsilon}})^{\frac{1}{2}} ((\lambda + \varepsilon)(\mu_{f^{-b}(z)}, \mu_{f^{-b}(z)})_{\frac{r}{\lambda - \varepsilon}})^{\frac{1}{2}} \\ & \leq (\lambda + \varepsilon) \frac{(\mu_{f^{-a}(z)}, \mu_{f^{-a}(z)})_{\frac{r}{\lambda - \varepsilon}} + (\mu_{f^{-b}(z)}, \mu_{f^{-b}(z)})_{\frac{r}{\lambda - \varepsilon}}}{2}. \end{aligned} \quad (6)$$

Moreover

$$\begin{aligned} & (\mu_{f^{-a}(z)} \circ f^{-a}, \mu_{f^{-b}(z)} \circ f^{-b})_r \\ & = \int \mu_{f^{-a}(z)} \circ f^{-a}(B_r(x)) \mu_{f^{-b}(z)} \circ f^{-b}(B_r(x)) dx \\ & = \int \int \int I_{\{s: |s-x| < r\}}(s) I_{\{t: |t-x| < r\}}(t) \\ & \quad d\mu_{f^{-a}(z)} \circ f^{-a}(s) d\mu_{f^{-b}(z)} \circ f^{-b}(t) dx \\ & \leq \int \int 2r I_{\{(s, t): |s-t| < 2r\}}(s, t) d\mu_{f^{-a}(z)} \circ f^{-a}(s) d\mu_{f^{-b}(z)} \circ f^{-b}(t) \\ & = \int \int 2r I_{\{(c, d): |\rho^{-1}(\dots c_{-2} c_{-1} a \rho_0(z)) - \rho^{-1}(\dots d_{-2} d_{-1} b \rho_0(z))| < 2r\}}(c, d) \\ & \quad d\mu_{\Sigma}(c) d\mu_{\Sigma}(d). \end{aligned} \quad (7)$$

If a and b are such that there exists a k such that either $a \in A_0$, $b \in A_1$ or $a \in A_1$, $b \in A_0$, then by Lemma 3.1 and (7) we get that

$$\begin{aligned} & \int_{-1}^1 (\mu_{f^{-a}(z)} \circ f^{-a}, \mu_{f^{-b}(z)} \circ f^{-b})_r \, dz \\ & \leq \iint 2rv\{z : |\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)) - \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))| < 2r\} \\ & \quad d\mu_\Sigma(c)d\mu_\Sigma(d) \\ & \leq 8r^2 \frac{1}{C_{\lambda,\varepsilon,m}\varepsilon}, \end{aligned}$$

and so since there are at most $2^{2(m+1)}$ pairs a, b such that either $a \in A_0$, $b \in A_1$ or $a \in A_1$, $b \in A_0$, we get that

$$\sum_{\substack{a \in A_0, b \in A_1 \\ \text{or} \\ a \in A_1, b \in A_0}} \int_{-1}^1 (\mu_{f^{-a}(z)} \circ f^{-a}, \mu_{f^{-b}(z)} \circ f^{-b})_r \, dz \leq \frac{8r^2}{C_{\lambda,\varepsilon,m}} 2^{2(m+1)}.$$

For the case that $a, b \in A_0$ or $a, b \in A_1$ we use (6) to estimate that

$$\sum_{\substack{a, b \in A_0 \\ \text{or} \\ a, b \in A_1}} (\mu_{f^{-a}(z)} \circ f^{-a}, \mu_{f^{-b}(z)} \circ f^{-b})_r \leq (\lambda + \varepsilon) 2^m \sum_a (\mu_{f^{-a}(z)}, \mu_{f^{-a}(z)})_{\frac{r}{\lambda - \varepsilon}}.$$

Now, (5) and the last two estimates implies

$$\begin{aligned} J(r) & \leq \frac{8}{C_{\lambda,\varepsilon,m}\varepsilon} + 2^{-2(m+1)}(\lambda + \varepsilon) 2^m \frac{1}{r^2} \sum_a \int_{-1}^1 (\mu_{f^{-a}(z)}, \mu_{f^{-a}(z)})_{\frac{r}{\lambda - \varepsilon}} \, dz \\ & = \frac{8}{C_{\lambda,\varepsilon,m}\varepsilon} + 2^{-2(m+1)} \frac{\lambda + \varepsilon}{(\lambda - \varepsilon)^2} 2^m \frac{1}{\left(\frac{r}{\lambda - \varepsilon}\right)^2} \sum_a 2^m \int_a (\mu_z, \mu_z)_{\frac{r}{\lambda - \varepsilon}} \, dz \\ & = \frac{8r^2}{C_{\lambda,\varepsilon,m}\varepsilon} + 2^{-2}(\lambda - \varepsilon)^{-2} \left(\frac{\lambda + \varepsilon}{\lambda - \varepsilon}\right) J\left(\frac{r}{\lambda - \varepsilon}\right). \end{aligned}$$

Hence

$$\begin{aligned} \frac{J(r)}{r^2} & \leq \frac{8}{C_{\lambda,\varepsilon,m}\varepsilon} \sum_{n=0}^{\infty} (2(\lambda - \varepsilon))^{-2n} \left(\frac{\lambda + \varepsilon}{\lambda - \varepsilon}\right)^n \\ & = \frac{8}{C_{\lambda,\varepsilon,m}\varepsilon} \frac{(\lambda - \varepsilon)^2}{(\lambda - \varepsilon)^2 - \frac{1}{4} \frac{\lambda + \varepsilon}{\lambda - \varepsilon}}. \end{aligned}$$

By (3) we thus have

$$\|\mu_{\lambda,\varepsilon,m}\|_2 \leq \frac{c_m}{\sqrt{\varepsilon}}$$

with

$$c_m = \sqrt{8 \frac{(\lambda - \varepsilon)^2}{(\lambda - \varepsilon)^2 - \frac{1}{4} \frac{\lambda + \varepsilon}{\lambda - \varepsilon}} \left(\frac{2}{\lambda + \varepsilon} - \frac{4(\lambda + \varepsilon)}{2^m - \lambda - \varepsilon} \right)^{-1}}.$$

By (1) we have

$$\|\nu_{\lambda, \varepsilon, m}\|_2 \leq 2 \|\mu_{\lambda, \varepsilon, m}\|_2 \leq \frac{c_m}{\sqrt{\varepsilon}}.$$

Letting $m \rightarrow \infty$ we find that

$$\|\nu_{\lambda, \varepsilon}\|_2 \leq \frac{c}{\sqrt{\varepsilon}}.$$

with

$$c = 4 \sqrt{\frac{(\lambda^2 - \varepsilon^2)(\lambda - \varepsilon)}{(\lambda - \varepsilon)^2 - \frac{1}{4} \frac{\lambda + \varepsilon}{\lambda - \varepsilon}}} < \frac{4\lambda\sqrt{\lambda}}{\sqrt{\lambda^2 - \frac{1}{4} - (2\lambda + 1)\varepsilon}}.$$

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