Definition. A periodic point $p$ is Lyapunov stable if for all $\varepsilon > 0$ there is a $\delta > 0$ s.t.

$$\|x - p\| < \delta \Rightarrow \|f^k(x) - f^k(p)\| < \varepsilon$$

for all $k \geq 0$.

The point $p$ is called attracting if it is Lyapunov stable and if there is a $\delta > 0$ s.t.

$$\lim_{k \to \infty} \|f^k(x) - f^k(p)\| = 0$$

for all $x$ with $\|x - p\| < \delta$.

In the previous example $0$ is not Lyapunov stable, but $f^k(x) \to 0$ for all $x$.

Definition. A periodic point $p$ is unstable if it is not Lyapunov stable.

It is repelling if there exists $r > 0$ s.t. for all $x$ with $0 < \|x - p\| < r$ there exists $k$ with $\|f^k(x) - f^k(p)\| > r$. 
Theorem. Let $I$ be an interval and $f : I \rightarrow I$. Suppose that $p$ is a fixed point and that $f$ is continuously differentiable in an open neighbourhood of $p$.

- $|f'(p)| < 1 \Rightarrow p$ is attracting
- $|f'(p)| > 1 \Rightarrow p$ is repelling.

Similarly for periodic points.

**Illustration:**

If $f(p) > 0$, then all $x_n$ are on one side of $p$.

If $f'(p) < 0$ then $x_n$ is alternating between the two sides of $p$.  

\[
0 < f'(p) < 1
\]

\[
\begin{align*}
(p, p) &= (p, f(p)) \\
y &= x \\
x \quad x_2 \quad x_1 \quad x_0 \\
\end{align*}
\]

\[
-1 < f'(p) < 0
\]

\[
\begin{align*}
(y, f(x)) \\
y &= f(x) \\
x \quad x \quad x_2 \quad x_1 \\
\end{align*}
\]
Proof

Since $f'$ is continuous, $f'(x) \neq f'(p)$ if $x$ is close to $p$.

Assume that $|f'(p)| < 1$. Then we can take $\delta > 0$ and $\lambda$ such that

$$|f'(x)| < \lambda < 1$$

for all $x \in [r-\delta, r+\delta]$.

Take $x_0 \in [r-\delta, r+\delta]$. Then

$$f(p) - f(x_0) = f'(x_0) (p - x_0)$$

for some $x_0$ between $x$ and $p$.

Letting $x_k = f^k(x_0)$, we then have

$$|p - x_1| = |f(p) - f(x_0)| \leq \lambda |p - x_0|.$$

Hence $x_1 \in [r-\delta, r+\delta]$ and in the same way

$$|p - x_2| \leq \lambda |p - x_1| \leq \lambda^2 |p - x_0|$$

$$\vdots$$

$$|p - x_k| \leq \lambda^k |p - x_0|.$$

Since $\lambda \in (0, 1)$, this shows that $p$ is attracting.

If $|f'(p)| > 1$, then in a similar way

$$|f'(x)| > \mu > 1$$

for $x \in [r-\delta, r+\delta]$ and

$$|p - x_k| \geq \mu^k |p - x_0| \quad (\text{as long as } x_k \in [r-\delta, r+\delta]).$$
Note that if \(|f'(p)| < 1\), then the proof shows that

\[ |x_k - p| \approx |f'(p)|^k |x_0 - p|. \]

If \(f'(p) \neq 0\), then \(x_n\) converges to \(p\) with an exponential speed \(|f'(p)|^k\), but if \(f'(p) = 0\), then \(x_n\) converges to \(p\) faster than any exponential speed.

**Definition.** \(p\) is a superattracting fixed point if \(f(p) = p\) and \(f'(p) = 0\).

Similarly for periodic points.

**Example.** The quadratic family

\[ f_a(x) = ax(1-x). \]

\[ f_a'(x) = 0 \iff x = \frac{1}{2}. \]

\[ f_a(\frac{1}{2}) = \frac{1}{2} \iff a = 2. \]

Hence \(f_a\) has a superattracting fixed point if and only if \(a = 2\).

**Example.** Newton–Raphson.

Fixed points are super-attracting. Fast convergence!
BIFURCATIONS FOR THE QUADRATIC FAMILY

TOMAS PERSSON

Let \( f_a(x) = ax(1-x) \) be defined on \([0,1]\). Below we plot \( y = f_a(x) \) along with \( y = f_a^2(x) \) for increasing values of \( a \).

There is always the fixed point \( x_0 = 0 \). For \( a > 1 \) there is also the fixed point \( x_1 = \frac{a-1}{a} \). These are the only fixed points.

Below we see that \( x_1 \) is an attracting fixed point (we have \( |f'_a(x_1)| < 1 \)) for the three values of \( a \) considered. We also see that there are no points of period 2, that is there are no fixed points of \( f_a^2 \) apart from the points that are fixed by \( f_a \).

However, when we increase \( a \), the modulus of the derivative at the fixed point \( x_1 \) will increase, and at \( a = 3 \) we will have \( |f'_a(x_1)| = 1 \). This is illustrated in the three figures below. We observe that if we increase \( a \) slightly over \( a = 3 \), then there will appear two new fixed points of \( f_a^2 \). These new period two points are attracting, and the old fixed point is now repelling.

In other words, when increasing \( a \) over the value \( a = 3 \), the attracting fixed point \( x_1 \) splits into one repelling fixed point, and two attracting period two points.

This process will continue if we further increase \( a \). The modulus of the derivative of \( f_a^2 \) at the period two points will increase, and at a certain value of \( a \) they will become repelling. In the same way as above they will then split into one repelling fixed point of \( f_a^2 \) (that is a repelling period two point of \( f_a \)) and two attracting fixed points of \( f_a^4 \) (that is attracting period four

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points of \( f_a \). The period of the “new” periodic points are two times as large as the period of the “original” periodic point. Therefore, this is called period doubling.

It is easy to check that the period doubling described above takes place at \( a = 1 + \sqrt{6} \approx 3.44949 \). This is illustrated in the pictures below, where we also display the graphs \( y = f^4_a(x) \).

This process will now continue as we increase \( a \) even further. We will experience repeated called period doubling, as described above. It turns out that the intervals between two consecutive parameter values with period doubling get smaller and smaller, and in fact the quotient between their lengths approaches the so called Feigenbaum constant \( 4.66920\ldots \). Hence they must accumulate at some value of \( a \), and this value is \( a = 3.54409\ldots \).

The picture below shows the attractor of \( f_a \) for different values of \( a \). (Here \( a \) is called \( r \), since I took the picture from wikipedia.)

At \( a \approx 3.82843 \) a period three orbit appears as shown below, where the graphs \( y = f_a(x) \) and \( y = f^3_a(x) \) are drawn. Try to find the corresponding part in the picture from wikipedia.
Conjugacies.

A metric on a set $X$ is a function $d : X \times X \to [0, \infty)$ s.t.

$$d(x, y) \geq 0, \quad d(x, y) = d(y, x),$$
$$d(x, y) = 0 \iff x = y,$$
$$d(x, z) \leq d(x, y) + d(y, z).$$

The metric measures the distance between points.

For instance, if $X$ is an interval, then $d(x, y) = |x - y|$ is a metric.

If $X$ is a set with a metric $d$, then $(X, d)$ is called a metric space.

If $(X, d_X)$ and $(Y, d_Y)$ are metric spaces, then $h : X \to Y$ is continuous if for any $x_0 \in X$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(x, x_0) < \delta \Rightarrow d(h(x), h(x_0)) < \varepsilon.$$

In words: $h$ maps nearby points in $X$ to nearby points in $Y$. 
Consider two dynamical systems $f : X \to X$ and $g : Y \to Y$, where $X$ and $Y$ are metric spaces.

A (topological) conjugacy from $f$ to $g$ is a continuous mapping $h : X \to Y$ which has a continuous inverse $h^{-1} : Y \to X$ and which satisfies

$$h \circ f = g \circ h.$$

Think of $h$ as a continuous change of variables.

Example

$$f_b(x) = b x^2 - 1,$$
$$g_a(y) = a y (y - 1).$$

Let $h(x) = \frac{1}{2} - \frac{b}{a} x$.

Then $h \circ f_b = g_a \circ h$ if

$$b = \frac{1}{4} a^2 - \frac{1}{2} a.$$

Proof: Compute $h \circ f_b(x)$ and $g_a \circ h(x)$ and check that they are equal if $b = \frac{1}{4} a^2 - \frac{1}{2} a$. 
Example. Let \( T(x) \) be the tent map on \([0,1]\) and \( f(y) = 4y(1-y) \).

Put \( h(x) = \sin^2\left(\frac{x}{2}\right) \). Then

\[ h \) is a conjugacy:

\[ h \circ T(x) = f \circ h(x) \quad \text{for all } x \in [0,1]. \]

Proof. \( f \circ h(x) = 4h(x)(1-h(x)) \)

\[ = 4 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) \]
\[ = \sin^2(x). \]

For \( 0 \leq x \leq \frac{1}{2} \):

\[ h \circ T(x) = h(2x) = \sin^2(2x) \]

For \( \frac{1}{2} \leq x \leq 1 \):

\[ h \circ T(x) = h(2-2x) \]
\[ = \sin^2(\pi-2x) = \sin^2(\pi x) \]

Corollary. All periodic points of \( f(x) = 4x(1-x) \) are repelling.

Proof. \( p \) is a periodic point of \( T \) with period \( n \) \( \Leftrightarrow h(p) \) is a periodic point of \( f \) of period \( n \).