Attractors

We consider a diff. eqn. $\dot{x} = F(x)$ in $\mathbb{R}^d$.

A trapping region is a compact set $U$ s.t.

$y(t, U) \subset \text{interior}(U)$ for $t \geq 0$.

The attracting set of a trapping region $U$ is the set

$$A = \bigcap_{t \geq 0} y(t, U).$$

$A$ is an attractor if it is an attracting set with no proper non-empty subset which is also an attracting set.

(i.e. $\emptyset \neq A' \subset A \Rightarrow A'$ is not an attracting set.)
An invariant set $S$ is topologically transitive if for any $x, y \in S$ and any open balls $B_1, B_2$ with $x \in B_1$ and $y \in B_2$, there is a $T \in O$ s.t.

$$\gamma(T, B_1) \cap B_2 \neq \emptyset.$$ 

Theorem. Let $A$ be the attracting set of $U$. Then

- $A$ is forward and backward invariant.
- $x \in U \Rightarrow \omega(x) \subset A$.
- If $x \in A$ is a hyperbolic fixed point, then $W^s(x) \subset A$.

(Similar for hyperbolic orbit.)
Proof. Let $s \in \mathbb{R}$. Then

$$\mathcal{Y}(s,A) = \bigcap_{t \geq 0} \mathcal{Y}(t+s, A)$$

$$= \bigcap_{t \geq s} \mathcal{Y}(t, A).$$

If $s < 0$, then

$$\mathcal{Y}(s,A) = \bigcap_{t \geq s} \mathcal{Y}(t, A) = \bigcap_{s \leq t < 0} \mathcal{Y}(t, A) \cap \bigcap_{t \geq 0} \mathcal{Y}(t, A)$$

$$\subset \bigcap_{t \geq 0} \mathcal{Y}(t, A) = \mathcal{Y}(0,A) = A.$$ 

If $s \geq 0$, then (since $\mathcal{Y}(s,A) \subset A$)

$$\mathcal{Y}(s,A) = \bigcap_{t \geq 0} \mathcal{Y}(t+s, A) =$$

$$= \bigcap_{t \geq 0} \mathcal{Y}(t, \mathcal{Y}(s,A))$$

$$\subset \bigcap_{t \geq 0} \mathcal{Y}(t, A) = A.$$ 

The other parts of the theorem should be rather clear.
Reparametrisation: A function \( \tau : \mathbb{R} \to \mathbb{R} \)
which is strictly increasing and s.t.
\[
\tau(t) \to \infty, \quad t \to \infty,
\]
\[
\tau(t) \to -\infty, \quad t \to -\infty.
\]
(This is a change of the \( t \) variable.)

The differential equation \( \dot{x} = F(x) \)
has sensitive dependence on initial conditions at \( x \) if there is an \( r > 0 \)
s.t. for any \( \delta > 0 \) there is \( y \) with
\[
\| y - x \| < \delta \quad \text{and for any reparam.} \quad \tau \quad \text{there is} \quad T > 0 \quad \text{s.t.}
\]
\[
\| y(\tau(T), y) - y(T, x) \| > r.
\]

Example \( \begin{cases} \dot{r} = 0 \\ \dot{\theta} = r \end{cases} \)
no sensitive dependence on initial conditions!
A nearby orbit will have
\[
\| y(T, y) - y(T, x) \| > r
\]
but not for every reparam.

Example \( \begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases} \) Has sensitive dep. on initial cond.
The orbits diverge from each other.
An attractor is called chaotic if it is transitive and has sensitive dependence on initial conditions when restricted to $A$. (This means that we have sensitive dep. on in cond. for any $x \in A$ and we can take $y \in A$ in the definition of sensitive dep. on in cond.)

A strange attractor is an attractor which has "complicated geometry". (No precise def.)

Note that there are other ways to define chaos!

An example of a strange and chaotic attractor: The Lorenz attractor.

Lorenz system

\[
\begin{align*}
\dot{x} &= -\sigma x + \sigma y \\
\dot{y} &= r x - y - xz \\
\dot{z} &= -bz + xy
\end{align*}
\]

$\sigma = 10, \ b = \frac{8}{3}, \ r = 28$. 
See computer graphics!

There are three fixed points:

1. The origin. One unstable and two unstable directions.
2. $P_+ = (\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1)$

Each of these has one unstable direction and they are stable foci in a surface.

Consider the Poincaré map of

$$\Sigma = \{ (x, y, z) : |x|, |y| \leq \sqrt{b(r-1)}; z = r-1 \}.$$
Because of the flatness of the attractor, we can approximate the Poincaré map by a one dimensional map \( f: \text{Interval} \rightarrow \text{Interval} \).

(look at \( \Sigma \) from one side, see picture.)
The one-dimensional Poincaré map is then something like

We can choose coordinates so that

\[ f: [-1, 1] \to [-1, 1]. \]

One can "observe" that \( f' > 1 \).

Because of \( f' > 1 \), we have sensitive dependence on initial condition.

Proof. Let \( \lambda > 1 \), s.t. \( f' > \lambda \).

Take \( x \) and \( y \) close, but \( x \neq y \).

Let \( x_n = f^n(x) \), \( y_n = f^n(y) \)

Then if \( x_n \) and \( y_n \) are on the same side of 0, we have

\[ |x_{n+1} - y_{n+1}| \geq \lambda |x_n - y_n|. \]

Hence, the distance between \( x_n \) and \( y_n \) will grow in \( n \), until \( x_n \) and \( y_n \) are on different sides of 0.
One can also prove that $[-1, 1]$ is a chaotic attractor of $f$. (It is transitive. See the book.)

Suppose $f' > \sqrt{2}$. The argument is the following. Consider a small interval $I \subset [-1, 1]$. If $0 \notin I$, then $|f(I)| > \sqrt{2} |I|$ where $|I|$ denotes the length of $I$. If $0 \in I$, then $f(I)$ consists of two intervals $f(I_-)$ and $f(I_+)$. 

We have $|f(I_-)| > \sqrt{2} |I_-|$ and $|f(I_+)| > \sqrt{2} |I_+|$. For both $f(I_-)$ and $f(I_+)$, 0 is not in $f(I)$ so 

$$|f(f(I_-))| > \sqrt{2} |f(I_-)| > 2 \cdot |I_-|$$

and 

$$|f(f(I_+))| > \sqrt{2} |f(I_+)| > 2 \cdot |I_+|$$

This implies that $f(f(I)) - f^2(I)$ consists of two intervals, at least one larger than $I$.
When we iterate \( f^k(I) \), \( k = 0, 1, 2, \ldots \), then for each \( k \), we have several intervals, but at least one interval which is getting larger and larger. This can be used to prove the transitivity of \( f \).

Lyapunov exponents.

Let \( x_0 \) be an initial condition and \( v_0 \) a vector. Put

\[
\begin{align*}
\dot{v}(t) &= \frac{d}{ds} f(t, x_0 + sv_0) \bigg|_{s=0} \\
&= D_{x_0} f(t, x_0) v_0.
\end{align*}
\]

Recall from lecture 2-3 that \( \dot{v} \) satisfies the differential equation

\[
\dot{v}(t) = DF_{x_0(t, x_0)} v(t).
\]

The limit (if it exists)

\[
\lambda(x_0, v_0) = \lim_{t \to \infty} \frac{\ln \| v(t) \|}{t}
\]

is called the Lyapunov exponent at \( x_0 \) in the direction \( v_0 \).