Preserved quantities

In a system of the form

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= F(x)
\end{align*}
\]

where \( x, y \in \mathbb{R}^d \), and \( F(x) = -\nabla V(x) = -\left( \frac{\partial V}{\partial x_1}(x), \frac{\partial V}{\partial x_2}(x), \ldots \right) \)

for some scalar function \( V \), the energy

\[ E(x, y) = \frac{1}{2} \| \dot{x} \|^2 + V(x) \]

\[ = \frac{1}{2} (\dot{x}, \dot{x}) + V(x) \]

is preserved:

\[ \dot{E}(x, y) = (\ddot{x}, \dot{x}) + (\nabla V(x), \dot{x}) \]

\[ = (\ddot{x} - F(x), \dot{x}) = 0 \]

Example. Pendulum

\[ V(x) = -gml \cos x. \]

Suppose \( gml = 1 \).

Then

\[ E(x, y) = \frac{1}{2} y^2 - g \sin x. \]
Level sets of \( E(x,y) = \frac{1}{2} y^2 - \cos x \)

A so-called heteroclinic orbit:

An orbit \( g(t,p) \) with

\[
\lim_{t \to +\infty} g(t,p) = q_1 \\
\lim_{t \to -\infty} g(t,p) = q_2
\]

and \( q_1 \) and \( q_2 \) are different fixed points.

If \( q_1 = q_2 \) and are fixed points, then the orbit is called a homoclinic orbit.

Example Duffin equation

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= x - x^3
\end{align*}
\]

Example Predator–prey

\[
\begin{align*}
\dot{x} &= x(a - by) \\
\dot{y} &= y(-c + ex)
\end{align*}
\]
A nonlinear centre is a fixed point which is surrounded by periodic orbits. Can be observed in the examples above.

(Notice that it is not enough to look at the linearised system to conclude that a nonlinear centre exists. See previous lecture.)

A system where energy is preserved is called conservative. If energy is not preserved, it is called dissipative.

Example: Damped pendulum

\[
\begin{align*}
    \dot{x} &= y \\
    \dot{y} &= -\sin x - by
\end{align*}
\]

The energy decreases along orbits.
Definition. Let $p$ be a fixed point of the system $\dot{x} = F(x)$, $x \in \mathbb{R}^d$. A differentiable function $L : \mathbb{R}^d \to \mathbb{R}$ is called a weak Lyapunov function of $p$ if there is an open set $U$, with $p \in U$ and s.t.

\begin{itemize}
  \item $L(x) > L(p)$ for all $x \in U$, $x \neq p$
  \item $L(x) = (\forall \delta, \dot{x}) = (\forall \delta, F(x)) \leq 0$
    for all $x \in U$.
\end{itemize}

The function $L$ is called a (strict) Lyapunov function if it is a weak Lyapunov function and

$L(x) < 0$ for all $x \neq p$, $x \in U$.

Example. Energy is a weak Lyapunov function for the clamped pendulum, but not a strict Lyapunov function.
Recall: A fixed point \( p \) is

- Lyapunov stable if for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) s.t.
  \[
  \| x_0 - p \| < \delta \Rightarrow \| g(t, x_0) - p \| < \varepsilon \quad \text{for all } t \geq 0.
  \]

- \( \omega \)-attracting if there is a \( \delta > 0 \) s.t. \( \omega(x_0) = \{ p \} \) for all \( x_0 \) with \( \| x_0 - p \| < \delta \)

- attracting if it is Lyapunov stable and \( \omega \)-attracting.

Theorem. Suppose that \( p \) is a fixed point of \( \dot{x} = F(x) \).

- If \( \lambda \) is a weak Lyapunov function of \( p \), then \( p \) is Lyapunov stable.
- If \( \lambda \) is a strict Lyapunov function of \( p \), and if \( \lambda_0 > \lambda(p) \) and the set
  \[
  U_{\lambda_0} = \{ x \in U : \lambda(x) \leq \lambda_0 \}
  \]
is compact, then
  \[
  U_{\lambda_0} \subset W^s(p)
  \]
  (i.e. \( x \in U_{\lambda_0} \Rightarrow \lim_{t \to \infty} g(t, x) = p \). )
Proof 1) Let \( \varepsilon > 0 \). Put
\[
V_K = \{ x \in \text{closure}(U) : \mathcal{L}(x) \leq K \} \cap \{ x : \| x - p \| \leq \varepsilon \}
\]
The sets \( V_K \) are compact and non-empty if \( K \geq \mathcal{L}(p) \).
We have
\[
\{ p \} = \bigcap_{K \geq \mathcal{L}(p)} V_K.
\]
There must exist a \( K_0 > \mathcal{L}(p) \) s.t.
\[
\{ p \} \subset V_{K_0} \subset \{ x : \| x - p \| < \varepsilon \}.
\]
Since \( \mathcal{L} \) is continuous, there is a \( \delta > 0 \) s.t.
\[
\{ x : \| x - p \| < \delta \} \subset V_{K_0}.
\]
Since \( \mathcal{L}(y(t, x_0)) \) cannot increase, the set \( V_{K_0} \) is forward invariant.
We get
\[
\| x_0 - p \| < \delta \Rightarrow x_0 \in V_{K_0} \Rightarrow y(t, x_0) \in V_{K_0} \text{ for all } t \geq 0 \Rightarrow y(t, x_0) \in \{ x : \| x - p \| < \varepsilon \} \text{ for all } t \geq 0.
\]
Hence \( p \) is Lyapunov stable.
2) The set $U_\infty$ is forward invariant.

Suppose that $x_0 \in U_\infty$ and that $\gamma(t, x_0) \not\to p, \ t \to \infty$. Since $U_\infty$ is compact and forward invariant, we have

\[
\emptyset \neq \omega(x_0) \subseteq U_\infty
\]

and $\omega(x_0) \neq \{ p \}$.

Take $q \in \omega(x_0)$, $q \neq p$. Then $\gamma'(q) < 0$. Take $\varepsilon > 0$ so small that

\[
\| x - q \| < \varepsilon \Rightarrow \gamma'(x) < \frac{1}{2} \gamma'(q).
\]

Since $q \in \omega(x_0)$, the orbit $\gamma(x_0, t)$ must enter the ball $\{ x : \| x - q \| < \frac{\varepsilon}{2} \}$ for arbitrary large $t$. Each time, the orbit spends at least a time $T > 0$ in the ball $\{ x : \| x - q \| < \frac{\varepsilon}{2} \}$. In total, the orbit spends infinite amount of time in $\{ x : \| x - q \| < \frac{\varepsilon}{2} \}$, and since $\gamma$ is uniformly bounded away from 0 in this ball, we must have $\gamma'(\gamma(t, x_0)) \to -\infty$ which is impossible.
Gradient Systems.

Suppose that $G : \mathbb{R}^d \to \mathbb{R}$ is two times continuously differentiable. (Then $\frac{\partial^2 G}{\partial x_i \partial x_j} = \frac{\partial^2 G}{\partial x_j \partial x_i}$.)

The system $\dot{x} = -\nabla G(x)$ is called a gradient system.

"Theorem" Let $p$ be a strict local minimum of $G$. Then $G$ is a strict Lyapunov function of $p$.

Counterexample in $\mathbb{R}$:

\[ \text{graph of } G_1 \]

$p$, global minimum of $G_1$.

points where $G_1' = 0$, which accumulate at the minimum. There is no open interval around $p$, where $G' = 0$ only in $p$. 
Theorem. Let $p$ be a strict local minimum of $G$. Then $G$ is a weak Lyapunov function of $p$.

Proof. We have
\[ \dot{G}(x) = \langle \nabla G(x), \dot{x}(t) \rangle = \]
\[ = -\langle \nabla G(x), \nabla G(x) \rangle \leq 0. \]
Since $p$ is a strict local minimum there exists an open set $U$, with $p \in U$ and
\[ G(q) > G(p) \text{ for all } q \in U, \]
$q \neq p$.

This proves that $G$ is a weak Lyapunov function.

Theorem. Suppose that $\nabla G(p) = 0$. Then all eigenvalues of the linearised system are real.

Proof. The matrix $D(-\nabla G)_p = \left[ -\frac{\partial^2 G}{\partial x_i \partial x_j} \right]_{i,j}$ is symmetric. \(\Rightarrow\) All eigenvalues are real.