Higher dimensional dynamical systems.

\[ X \subset \mathbb{R}^d \text{ and } f : X \to X \]

Suppose that \( p \) is a fixed point, \( p = f(p) \).

If \( f \) is differentiable, we have locally

\[
\mathcal{J}(x) \approx f(p) + \mathcal{D}f_p(x-p) \\
= p + \mathcal{D}f_p(x-p).
\]

So \( f(x) - p \approx \mathcal{D}f_p(x-p) \).

One might suspect that locally at \( p \)
\( f \) has a similar behaviour as

\[
g(x) = Ax
\]

at \( 0 \), where \( A = \mathcal{D}f_p \).

We therefore want to understand the linear dynamical system \( g(x) = Ax \).

This is very similar to the case of continuous time.

Consider for simplicity the case in \( \mathbb{R}^2 \) with 2 different eigenvalues \( \lambda_1 \) and \( \lambda_2 \).
If $\lambda_1, \lambda_2 \in \mathbb{R}$:
- Stable node if $|\lambda_1|, |\lambda_2| < 1$.
- Unstable node if $|\lambda_1|, |\lambda_2| > 1$.
- Saddle node if $|\lambda_1| < 1 < |\lambda_2|$.

If $\lambda_1, \lambda_2 \notin \mathbb{R}$:
- Stable focus if $|\lambda_1|, |\lambda_2| < 1$.
- Unstable focus if $|\lambda_1|, |\lambda_2| > 1$.
- Linear centre if $|\lambda_1| = |\lambda_2| = 1$.

Recall the definition of Lyapunov stable, unstable, attracting, and repelling:
- $p$ is Lyapunov stable if for all $r > 0$ there is a $\delta > 0$ s.t. $d(x, p) < \delta \Rightarrow d(f^n(x), p) < r$ for all $n \geq 0$.
- $p$ is unstable if it is not Lyapunov stable.
- $p$ is attracting if $p$ is Lyapunov stable and there is a $\delta > 0$ s.t. $d(x, p) < \delta \Rightarrow f^n(x) \to p, \quad n \to \infty$.
- $p$ is repelling if there is $r > 0$ s.t. $x \neq p \Rightarrow d(f^n(x), p) > r$ for some $n \geq 0$. 
Theorem \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \), \( C^2 \).

Let \( p \) be a fixed point and \( \lambda_1, \ldots, \lambda_d \) the eigenvalues of \( Df_p \) (counted with multiplicity).

If \( |\lambda_j| < 1 \) for all \( j \), then \( p \) is attracting.

If \( |\lambda_j| > 1 \) for all \( j \), then \( p \) is repelling.

If \( |\lambda_j| > 1 \) for some \( j \), then \( p \) is unstable.

This follows from the following theorem.

Theorem. Suppose that \( f \) is \( C^1 \) and \( \det(Df_p) \neq 0 \), \( f(p) = p \).

Then \( f \) is topologically conjugated to its linearisation in some open set around \( p \).

If \( p \) is a hyperbolic fixed point (\( |\lambda_j| \neq 1 \) for all \( j \)), then the conjugacy is \( C^1 \).

This is called the Hartman–Grobman or Grobman–Hartman theorem.
We sketch the proof in a special case. See Hartman's original article for a full proof. (Click on course homepage.)

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and assume that

\[
D f_p = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \text{ where } |a| < 1, \quad |b| > 1.
\]

We want to prove that there is a small ball \( B(p,r) \) and an invertible function \( R(x,y) = \begin{bmatrix} U(x,y) \\ V(x,y) \end{bmatrix} \) such that

\[
R \circ f = A \circ R,
\]

where \( A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \). We will construct \( R \) by successive approximations.

Write

\[
f([x]) = \begin{bmatrix} ax + X(x,y) \\ by + Y(x,y) \end{bmatrix}
\]

where \( X(x,y), Y(x,y) = o(1(|x,y|)) \).

Let \( \delta, \theta \in (0,1) \).

Take \( r > 0 \) s.t.

\[
|X(x,y)|, |Y(x,y)| \leq K_0 |x,y| \text{ when } |x,y| < r
\]

where \( K_0 \) satisfies \( \frac{(|b| + K_0)^\delta}{|b|} < \theta \).
We have
\[ U(ax + X(x,y), by + Y(x,y)) = aU(x,y) \]
\[ V(ax + X(x,y), by + Y(x,y)) = bV(x,y). \]
Consider \( V \). \( U \) is treated in a similar way.) Set
\[ V_0(x,y) = y, \]
\[ V_n(x,y) = \frac{1}{b} V_{n-1}(ax + X, by + Y). \]
Then \( V_1(x,y) = y + \frac{1}{b} Y \) and
\[ V_1 - V_0 = \frac{1}{b} Y, \]
\[ |V_1 - V_0| = \frac{1}{|b|} |Y| \leq K_1 \Theta l(x,y). \]
where \( K_1 = \frac{K_0}{|b| |b|}. \)
Let \( \tilde{V}_n = V_n - V_{n-1}. \) Then
\[ |\tilde{V}_1(x,y)| \leq K_1 \Theta l(x,y). \]
Suppose that \( |\tilde{V}_n| \leq K_1 \Theta^n l(x,y) \) for some \( n. \) Then
\[ |\tilde{V}_{n+1}(x,y)| = \left| \frac{1}{b} \tilde{V}_n(ax + X, by + Y) \right| \]
\[ \leq \frac{1}{|b|} K_1 \Theta^n l(ax + X, by + Y) \]
\[ \leq K_1 \Theta^n \frac{(|b| + K_0)}{|b|} l(x,y). \]
Hence \( |\tilde{V}_{n+1}| \leq K_1 \theta^{n+1} |(x,y)|^g \), since \( K_0 \) is such that \( \frac{(1.51 + K_0)}{1.51} < \theta \).

This proves that \( \tilde{V}_n \to 0 \) uniformly and \( \sum \tilde{V}_n \) is summable. Hence \( V_n \to V \) uniformly, where \( V \) is a continuous function.

One also has to prove that \( V \) is invertible.
Stable and unstable manifolds.

Let $p$ be a fixed point. The stable manifold of $p$ is

$$W^S(p) = \{ x : f^n(x) \to p, \ n \to \infty \}$$

If $f$ is invertible, we can define the unstable manifold:

$$W^U(p) = \{ x : f^{-n}(x) \to p, \ n \to -\infty \}$$

If $\delta > 0$, then we define the local stable and unstable manifolds.

$$W^S_\delta(p) = \{ x : d(f^n(x), p) < \delta \text{ for all } n \geq 0 \text{ and } f^n(x) \to p, \ n \to \infty \}$$

$$W^U_\delta(p) = \{ x : d(f^{-n}(x), p) < \delta \text{ for all } n \leq 0, \text{ and } f^{-n}(x) \to p, \ n \to -\infty \}$$

If $f$ is $C^1$, then $f$ is locally conjugated to its linearisation. The local stable and unstable manifolds of the linearisation are clearly smooth "curves", hence $W^S_\delta(p)$ and $W^U_\delta(p)$ are "curves" if $\delta$ is sufficiently small. (Curves if 1-dimensional, otherwise manifolds.)
Stable manifold theorem.

Suppose that \( f \) is \( C^r \) and that \( p \) is a hyperbolic fixed point. Then, if \( \delta > 0 \), the local stable and unstable manifolds, \( W^s_\delta(p) \) and \( W^u_\delta(p) \) are \( C^r \) manifolds, if \( \delta \) is sufficiently small.

(If \( p \) is not hyperbolic, then there is also a so called centre manifold.)

When \( p \) is hyperbolic we have

\[
W^s_\delta(p) = \{ x : d(f^n(x), p) < \delta \text{ for all } n \geq 0 \}
\]

\[
W^u_\delta(p) = \{ x : d(f^n(x), p) < \delta \text{ for all } n \leq 0 \}
\]

and

\[
W^s(p) = \bigcup_{n=1}^{\infty} f^{-n}(W^s_\delta(p))
\]

\[
W^u(p) = \bigcup_{n=1}^{\infty} f^n(W^u_\delta(p))
\]
Example \( \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) (the 2-torus).

Define \( f: \mathbb{T}^2 \to \mathbb{T}^2 \) by

\[
f([x], [y]) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{mod 1.}
\]

\[
= A
\]

This is sometimes called Arnold's cat map (because you can use it to transform cats).

We have \( \det A = 1 \), so \( A^{-1} \) exists and is an integer matrix with \( \det A^{-1} = 1 \).

Hence \( [x] \mapsto A[x] \) preserves area.

This implies that \( f \) is invertible and preserves area.

The eigenvalues of \( A \) are \( \frac{3 \pm \sqrt{5}}{2} \) with eigenvectors \( \begin{bmatrix} 2 \\ \sqrt{5} - 1 \end{bmatrix}, \begin{bmatrix} 15 - 1 \\ -2 \end{bmatrix} \).
Hence, the local unstable manifold of \( p = [0] \) is a line with tangent vector \( [\frac{2}{5} \, -1] \). Since \( \frac{\sqrt{5} - 1}{2} \) is not rational, \( W^u(p) \) is a dense curve which wraps around \( \mathbb{T}^2 \) infinitely many times.

Similarly for \( W^s(p) \).

Suppose that \( p \) is a fixed point s.t. \( W^s(p) \) intersects \( W^u(p) \). Then there is a "very" complicated dynamical behaviour. This was already observed by Henri Poincaré.

Picture!