Attractors and stuff.

Definition. \( f : X \to X \).

A point \( q \in X \) is an \( \omega \)-limit point of \( p \in X \) if \( \) for any \( \varepsilon > 0 \) there are infinitely many \( n \) such that
\[
d(f^n(p), q) < \varepsilon.
\]
The set of \( \omega \)-limit points of \( p \) is denoted by \( \omega(p) \).

If \( q \in \omega(p) \), then there is a sequence \( N_k \to \infty, k \to \infty \) such that
\[
f^k(p) \to q \text{ as } k \to \infty.
\]

Theorem. For the doubling map, the tent map, or \( f(x) = 4x(1-x) \), there are points \( p \in [0, 1] \) with
\[
\omega(p) = [0, 1].
\]
(We have seen before that there are points with dense orbit for these maps. Take \( p \) as such a point.)
Theorem. Consider the rotation \( R_x : [0,1) \to [0,1) \)
\[ R_x(\alpha) = \alpha + \alpha \mod 1 \]
If \( \alpha \in \mathbb{Q} \), then every \( x \) is a periodic point.
If \( \alpha \notin \mathbb{Q} \), then \( \omega(x) = [0,1) \)
for every \( x \).

Proof. If \( \alpha = \frac{p}{q} \in \mathbb{Q} \), with \( p, q \in \mathbb{Z} \),
then \( R_x^q(\alpha) = \alpha + q \alpha \mod 1 \)
\[ = \alpha + q \frac{p}{q} \mod 1 = \alpha + p \mod 1 \]
\[ = \alpha. \]
Hence every \( x \) is periodic.

Assume that \( \alpha \notin \mathbb{Q} \).

We use the following theorem by Dirichlet.

Theorem (Dirichlet): For any \( \alpha \), there is \( q < \alpha \)
and \( p \) s.t.
\[ |\alpha - \frac{p}{q}| < \frac{1}{q^2}. \]

Proof. See for instance "Topics in analysis", Thm 1.3.
In particular, for arbitrary large $Q$, there is $p,q$ s.t.

$$|x - \frac{p}{q}| < \frac{1}{q^2}$$

and we can find arb. large $q$ s.t. the above is true for some $p$. (This can also be concluded using continued fractions.)

Let $\varepsilon > 0$. Take $p,q$ s.t.

$$|x - \frac{p}{q}| < \frac{1}{q^2} < \varepsilon \cdot \frac{1}{q}.$$

Then $(q \cdot x \mod 1) < \varepsilon$ or

$$1 - (q \cdot x \mod 1) < \varepsilon.$$ 

Since $x \notin \mathbb{Q}$ we have $q \cdot x \mod 1 \neq 0$.

Assume that $0 < (q \cdot x \mod 1) < \varepsilon$,

the other case is similar.

This can be pictured as in the figure. (Think of $[0,1]$ as the circle.)

Hence $0, q \cdot x$ and $2q \cdot x$ lies on the circle as in this figure:

(Since $2q \cdot x = R_{x}^{2q}(0) = R_{x}^{q}(R_{x}^{q}(0))$. )
The distances between $0$ and $R_\alpha^n(0)$; $R_\alpha^3(0)$ and $R_\alpha(0); R_\alpha^2(0)$; $R_\alpha(0)$ and $R_\alpha(0); ...$ are the same since $R_\alpha$ is a rotation. These distances are less than $\varepsilon$.

Hence, the orbit of $0$ is dense as $\varepsilon > 0$ is arbitrary. It follows that $\omega(0) = [0, 1]$.

Since $R_\alpha$ is a rotation, we have

$$R_\alpha^n(x) = R_\alpha^n(0)$$

It follows that $\omega(x) = R_\alpha(\omega(0)) = [0, 1]$.

**Definition** $f : X \to X$. A set $V \subseteq X$ is called a trapping region if $V$ is closed and bounded and if

$$\text{closure}(f(V)) \subseteq \text{interior}(V).$$

**Example** If $f : [0, 1] \to [0, 1]$, $p$ is a fixed point with $f(p)$ if $f'(p) < 1$ and $f'(p) > 0$, then $[p - \delta, p + \delta]$ is a trapping region if $\delta$ is sufficiently small.
Definition. A set $A$ is called attracting if there is a trapping region $V$ s.t.

$$A = \bigcap_{j=0}^{\infty} f^j(V).$$

If $A$ is attracting with trapping region $V$, then $f(V) \subseteq V$.
Hence $f^2(V) \subseteq f(V) \subseteq V$ and so on. Therefore,

$$f(A) = \bigcap_{j=1}^{\infty} f^j(V) = \bigcap_{j=0}^{\infty} f^j(V) = A.$$  

In words: $A$ is forward invariant ($f(A) = A$).

Definition. $A$ is an attractor if it is an attracting set and no proper subset of $A$ is an attracting set.

$A$ is a transitive attractor if it is an attractor and there is a point in $A$ with orbit which is dense in $A$. 
Definition. A forward invariant set $E$ is called chaotic if $f$ restricted to $E$ is transitive and has sensitive dependence on initial conditions.

A chaotic attractor is an attractor which is chaotic.

Theorem. Let $f : [0, 1] \rightarrow [0, 1]$ be the doubling map, the tent map or $f(x) = 4x(1-x)$. Then $[0, 1]$ is a chaotic attractor.
Lyapunov exponents.

Suppose that \( X \subset \mathbb{R} \), and that \( f : X \to X \) is differentiable in at least some points of \( X \).

We define Lyapunov exponents in a similar way as for flows. As for flows, the Lyapunov exponent (if it exists) measures the exponential speed with which two nearby points are repelled or attracted to each other.

**Definition.** If \( f \) is differentiable along the orbit of \( p \), then the Lyapunov exponent at \( p \) is

\[
\lambda(p) = \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(p)|
\]

if the limit exists.

Since \((f^n)'(p) = f'(f^{n-1}(p)) \cdot f'(f^{n-2}(p)) \cdots f'(p)\)
we have

\[
\lambda(p) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(f^k(p))|.
\]
Note the following.

If \( l(p) \) exists, then
\[ l(T^k(p)) = l(p) \text{ for any } k \geq 0. \]

If \( z^p \) is a periodic point and \( f \) is differentiable at all points in the periodic orbit, then \( l(p) \) exists.

If \( q \) is a fixed point,
\[ f^n(p) \to q \] (and \( f' \) is sufficiently regular) then \( l(p) = l(q) \).