Symbolic Dynamics.

Theorem. Let \( f : I \to I \) be a continuous mapping on an interval \( I \). Suppose that \( J_0, J_1, \ldots, J_n \) are non-empty closed intervals such that

\[ J_{j+1} \subset f(J_j) \quad \text{for } j=0,1,\ldots,n. \]

Then there is an \( x \) s.t.

\[ f^j(x) \in J_j \quad \text{for } j=0,1,\ldots,n. \]

If moreover \( J_0 = J_n \), then \( x \) can be taken such that \( f^n(x) = x \).

Proof. The condition \( J_{j+1} \subset f(J_j) \) implies that whenever \( A \subset J_{j+1} \), there is \( B \subset J_j \) s.t. \( f(B) = A \).

Hence, there is \( K_{n-1} \subset J_{n-1} \) s.t. \( f(K_{n-1}) = J_n \). Hence there is \( K_{n-2} \subset J_{n-2} \) s.t. \( f(K_{n-2}) \subset K_{n-1} \) and \( f^2(K_{n-2}) = J_n \).

Continuing, we get \( K_0 \subset J_0 \) s.t. \( K_0 \subset J_0, f(K_0) \subset J_1, f(K_0) \subset J_2, \ldots, f^n(K_0) \subset J_n \).

We can take any \( x \in K_0 \).
We have 
\[ K_0 = \bigcap_{j=0}^n f^{-j}(I_j). \]

The intersection is nonempty and every \( f^{-j}(I_j) \) is compact since \( f \) is continuous.

Suppose that \( I_0 = I_n \). We define an infinite sequence of intervals \( I_j \) by

\[ I_0, I_1, \ldots, I_n = I_0, I_{n+1} = I_1, I_{n+2} = I_2, \ldots, I_{2n} = I_0, \ldots \]

Then \( \bigcap_{j=0}^m f^{-j}(I_j) \) is nonempty and compact for each \( m \), and therefore \( \bigcap_{j=0}^\infty f^{-j}(I_j) \) is nonempty.

Take \( x \in \bigcap_{j=0}^\infty f^{-j}(I_j) \).

Example: \( f(x) = 4x(1-x) \) def. on \([0,1]\).

Let \( I_0 = [0, \frac{1}{2}], I_1 = [\frac{1}{2}, 1] \). Then

\( I_0, I_1 \subset f(I_0) \) and \( I_0, I_1 \subset f(I_1) \).

The theorem implies that given \( a_0, a_1, \ldots, a_n \in \{0,1\} \), there is an \( x \) s.t.

\[ f^{J_j}(x) \in I_{a_j} \quad \text{for} \quad j = 0, 1, \ldots, n. \]
Suppose that $f: I \rightarrow I$

where $f$ is continuous and $I$ is an interval.

Write $I = \bigcup_{j=1}^{N} I_j$, where $I_j$ are closed intervals that only intersect at the boundary. (We call this a partition of $I$.)

Let $S = \{I_1, \ldots, I_N\}$. The transition graph is the graph with vertices in $S$ and with an edge $i \rightarrow j$ if and only if $I_j \subseteq f(I_i)$.

Example of a transition graph.
One sometimes calls $S$ an alphabet and a sequence $a_0, a_1, \ldots, a_n$ with $a_j \in S$ is called a word. A word is allowed if it corresponds to a path in the transition graph.

Let

$$I_{a_0, \ldots, a_n} = \left\{ x : f^j(x) \in I_{a_j}, \ j = 0, 1, \ldots, n \right\}$$

$$= \bigcap_{j=0}^{n} f^{-j}(I_{a_j}).$$

Then $I_{a_0, \ldots, a_n}$ is non-empty if $a_0, \ldots, a_n$ is allowed.
The above also holds for infinite words $a_1, \ldots, a_n, a_{n+1}, \ldots$.

**Theorem (Li and Yorke)**

Suppose that $f : I \to I$ is continuous. Assume that $x_0$ is such that

$$f^3(x_0) \leq x_0 < f(x_0) < f^2(x_0)$$

or

$$f^3(x_0) \geq x_0 > f(x_0) > f^2(x_0).$$

Then $f$ has a point of period $n$ for each $n \in \mathbb{N}$. 

Remark. The condition is satisfied if there is a point with period 3.

Proof. Assume that

\[ f^3(x_0) \leq x_0 < f(x_0) < f^2(x_0). \]

Let \( I_0 = [x_0, f(x_0)], I_1 = [f(x_0), f^2(x_0)]. \)

Then \( I_1 \subset f(I_0) \) and

\[ I_0, I_1 \subset f(I_0). \]

We get the transition graph

\[ 0 \rightarrow 1 \rightarrow 0 \]

In this graph there are periodic paths of any period.
Define the Sharkovskii ordering $\triangleright$ on $\mathbb{N}$ by

\[3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \ldots\]
\[2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright 2 \cdot 9 \triangleright \ldots\]
\[2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright 2^2 \cdot 9 \triangleright \ldots\]
\[2^3 \cdot 3 \triangleright 2^3 \cdot 5 \triangleright \ldots\]
\[\ldots \triangleright 2^4 \cdot 3 \triangleright 2^4 \cdot 5 \triangleright 2^4 \cdot 7 \triangleright 2^4 \cdot 9 \triangleright \ldots\]

**Theorem (Sharkovskii)**

Suppose that $f : I \rightarrow I$ is continuous and that $f$ has a point of period $k$. Then, if $k > n$, $f$ has a point of period $n$.

We do not look at the proof, but it is elementary.
Doubling map.

\[ D(x) = 2x \mod 1 \text{ defined on } [0, 1]. \]

Let \( I_0 = [0, \frac{1}{2}] \) and \( I_1 = [\frac{1}{2}, 1] \).

Suppose that \( x \) is s.t.
\[ f^i(x) \in I_{aj} \quad \text{for } j = 0, 1, ... \]

Then
\[ x = \sum_{j=0}^{\infty} \frac{a_j}{2^{j+1}}. \]

This is the binary expansion of \( x \).

If \( X \) is a metric space and \( f : X \to X \), then \( f \) is transitive (topologically transitive) if there is a point \( x \in X \) s.t.

\[ \mathcal{O}(x) = \{ x, f(x), f^2(x), ... \} \]

is dense in \( X \).

**Theorem** The doubling map is transitive.
Rotations on circles
\[ R_\alpha(x) = x + \alpha \mod 1 \]
(Identify the circle with \([0, 1]\).)
Then \( R_\alpha \) is transitive if and only if \( \alpha \) is an irrational number.

Shift spaces.
Let \( \Sigma_2 \) be the set of sequences \( a_0, a_1, \ldots \) such that \( a_j \in \{0, 1\} \).
Define \( \sigma : \Sigma_2 \to \Sigma_2 \) by
\[ \sigma(a_0, a_1, \ldots) = a_1, a_2, a_3, \ldots \]
\( \sigma \) is called the left shift.
Define \( \pi : \Sigma_2 \to [0, 1] \) by
\[ \pi(a_0, a_1, \ldots) = \sum_{j=0}^{\infty} \frac{a_j}{2^{j+1}}. \]
Then \( \pi \sigma = D \pi \), where \( D \) denotes the doubling map \( D(x) = 2x \mod 1 \).

It is almost true that \( \pi \) is a conjugacy from \( \sigma \) to \( D \), but this is not quite true: \( \pi \) is not one-to-one.

For instance
\[
\pi(1,0,0,...) = \frac{1}{2} \\
\pi(0,1,1,1,...) = \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2}
\]

If \( \pi(a_0,a_1,...) = \pi(b_0,b_1,...) \) then either \( a_0,a_1,... = b_0,b_1,... \) or one sequence ends with only zeroes and the other one ends with only ones, (as in the example above)
$\beta$-transformations

Let $\beta > 1$ and define

$$T_\beta(x) = \beta x \mod 1 \text{ on } [0, 1].$$

If $\beta = 2$, then we get the doubling map.

Similarly as for the doubling map we can define a shift space $\mathcal{S}_\beta$ of certain sequences $a_0, a_1, \ldots$ with $a_j \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$, and a mapping $T_\beta : \mathcal{S}_\beta \to [0, 1]$.

$\mathcal{S}_\beta$ are those sequences $a_0, a_1, \ldots$ such that there is an $x \in [0, 1]$ with

$$T_\beta^j(x) \in I_{a_j}$$

where $I_{a_j} = \left[ \frac{a_j}{\beta^j}, \frac{a_j + 1}{\beta^j} \right] \cap [0, 1]$.

$$T_\beta^x \text{ for } 2 < \beta < 3$$

$I_0$ $\xrightarrow{T_\beta}$ $I_1$ $\xrightarrow{T_\beta}$ $I_2$ $\xrightarrow{T_\beta}$ $I_3.$
The mapping $\tau_\beta$ is given by

$$\tau_\beta(a_0, a_1, \ldots) = \sum_{j=0}^{\infty} \frac{a_j}{\beta^{j+1}}$$

For instance, if $\beta = \frac{1+\sqrt{5}}{2}$, then $\Sigma_\beta$ are those sequences $a_0, \ldots \in \Sigma$ such that

$$a_j = 1 \Rightarrow \text{either } a_{j+1} = 0 \text{ or } a_{j+2}, a_{j+3}, \ldots = 1, 0, 0, 0, \ldots$$

Sensitive dependence on initial conditions.

**Definition** If $X \to X$, $(X, d)$ is a metric space. We say that $f$ has sensitive dep. on ini. cond. if for some $R > 0$ the following is true: For any $x$ and $\delta > 0$ there is a $y$ with $d(x, y) < \delta$ and

$$d(f^k(x), f^k(y)) \geq R$$

for some $k$. 


f is called expansive if for any \( x \) and \( y \neq x \), there is a \( k \) such that
\[
d(f^k(x), f^k(y)) \geq r.
\]

Clearly, the doubling map and the tent map are expansive, and they have sensitive dependence on initial conditions.

Since \( f(x) = 4x(1-x) \) is conjugated to the tent map, \( f(x) = 4x(1-x) \) is also expansive and has sensitive dep. on ini. cond.