Instructions. Give solutions to at most 5 problems. It is not allowed to get help from other human beings, but it is allowed to use pen and paper, books, computers, et cetera. Please hand in your solutions not later than November 9.

1. Consider the system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= x(x^2 - 1), \\
\ddot{x} &= x(x^2 - 1).
\end{align*}
\]

Find all fixed points and their stable and unstable manifolds. Are there any homoclinic or heteroclinic orbits?

2. Consider the system

\[
\begin{align*}
\dot{x} &= -y + xy^2, \\
\dot{y} &= x - x^2y^2.
\end{align*}
\]

Find all fixed points and describe their type. Draw a local phase portrait for each fixed point. Are the fixed points Lyapunov stable, unstable, \(\omega\)-attracting, attracting, \ldots?

3. Consider the system

\[
\begin{align*}
\dot{x} &= x^2(2y^3 - 1), \\
\dot{y} &= -y^2(x^2 - 1),
\end{align*}
\]

in the first quadrant, \(x, y > 0\). Show that every orbit is either a fixed point or a periodic orbit.

Hint: Find a function which is constant along the orbits.

4. Consider a differential equation \(\dot{x} = F(x)\) in \(\mathbb{R}^2\). Assume that \(\gamma\) is a periodic orbit such that there are no other periodic curves inside \(\gamma\). Prove that it is impossible that there are exactly two different fixed points \(p\) and \(q\) inside \(\gamma\) and that both \(p\) and \(q\) are attracting.

Is it possible to have two or more fixed points inside \(\gamma\), if they are only required to be Lyapunov stable?
5. Consider the Lorenz system

\[
\begin{align*}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= r x - y - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\]

where \( \sigma = 10, \ b = \frac{8}{3} \) and \( r = 28 \). Write a computer program that estimates the largest Lyapunov exponent of an initial point \((x_0, y_0, z_0)\). How does the Lyapunov exponent depend on the initial point? How does it vary if for instance the parameter \( \sigma \) is changed?

Comment: It is difficult to get precise numerics. Don’t bother about getting precise results.

6. Consider the system

\[
\begin{align*}
\dot{x} &= ax - y + x^2y, \\
\dot{y} &= x + ay - x^3 + y^3.
\end{align*}
\]

where \( a \in \mathbb{R} \) is a parameter. Show that there are values of \( a \) close to \( a = 0 \) for which the system has a periodic orbit around the origin.

Hint: Andronov–Hopf bifurcation.