Matristeori VT2010
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Kapitel 1– Kapitel 2

Matrices - Conventions and Notations

$a_{ij}$ is the element at the intersection of $i$–th row and $j$-th column

$$ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}. $$

Short notation is

$$ A = (a_{ij}), $$

Matrix of size $3 \times 4$

$$ A = \begin{pmatrix} 1 & 4 & 9 & 16 \\ 2 & 8 & 18 & 32 \\ 3 & 12 & 27 & 48 \end{pmatrix} $$
Square matrix of size 5 (size $5 \times 5$)

$$A = \begin{pmatrix}
1 & 4 & 9 & 16 & 25 \\
2 & 8 & 18 & 32 & 50 \\
3 & 12 & 27 & 48 & 75 \\
4 & 16 & 36 & 64 & 100 \\
5 & 20 & 45 & 80 & 125
\end{pmatrix}.$$  

$a_{32} = 12$

Ser ni något märkligt i värden på matriseelementen i denna matris?
\[ I = (\delta_{ij}) \]

Kronecker symbol

\[ \delta_{ij} = \begin{cases} 
1, & \text{if } i = j; \\
0, & \text{if } i \neq j.
\end{cases} \]

For square matrices this is **identity matrix** \( I \)

For \( n = 5 \)

\[
I = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

For the size \( 2 \times 4 \) it is different:

\[
(\delta_{ij}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]
A matrix of size $m \times n$

**transposed matrix** $B = A^T$ is of size $n \times m$

with $b_{ij} = a_{ji}$.

$$B = A^T = \begin{pmatrix}
a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\
a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\
a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn}
\end{pmatrix}$$

**OBS!** Every row becomes a column and vice versa

**Column vectors** of size $n = \text{The matrices}$

$X$ of size $n \times 1$

**Shorter one-index notations:**

$$X = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}.$$

**Row vectors** = matrices of size $1 \times n = \text{transposed column vectors}$ $Y = X^T$.

$$Y = (y_1, y_2, \ldots, y_n)$$
Block matrices.

\[
A = \begin{pmatrix}
1 & 4 & 9 & 16 & 25 \\
2 & 8 & 18 & 32 & 50 \\
3 & 12 & 27 & 48 & 75 \\
4 & 16 & 36 & 64 & 100 \\
5 & 20 & 45 & 80 & 125
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
X & Y \\
Z & T
\end{pmatrix},
\]

where

\[
X = \begin{pmatrix}
1 & 4 & 9 \\
2 & 8 & 18 \\
3 & 12 & 27
\end{pmatrix},
Y = \begin{pmatrix}
16 & 25 \\
32 & 50 \\
48 & 75
\end{pmatrix},
\]

\[
Z = \begin{pmatrix}
4 & 16 & 36 \\
5 & 20 & 45
\end{pmatrix},
T = \begin{pmatrix}
64 & 100 \\
80 & 125
\end{pmatrix}.
\]
Matrix $A$ of size $m \times n$ is also a block matrix of $n$ columns:

$$A = (A_1 \ A_2 \ \ldots \ A_n),$$

$$A_j = \begin{pmatrix}
    a_{1j} \\
    a_{2j} \\
    a_{3j} \\
    \ldots \\
    a_{mj}
\end{pmatrix}$$

$j$–th column (-vector) of $A$.

Row representation:

$$A = \begin{pmatrix}
    A_1 \\
    A_2 \\
    \ldots \\
    A_m
\end{pmatrix}$$

with

$$A_i = (a_{i1} \ a_{i2} \ \ldots \ a_{in}),$$
More careful notation:
Column representation

\[ A = (A_1 \ A_2 \ \ldots \ A_n) \]

Row representation

\[ A = \begin{pmatrix} A_1. \\ A_2. \\ \ldots \\ A_m. \end{pmatrix} \]
How to transpose a block matrix?

In a natural way, but also transpose the blocks

\[
\begin{pmatrix}
X & Y \\
Z & U
\end{pmatrix}^T =
\begin{pmatrix}
X^T & Z^T \\
Y^T & U^T
\end{pmatrix}.
\]
$E_{ij}$ matrix with all elements zero except one.

At the intersection of row $i$ and column $j$ there is a $1$ instead of a $0$.

Totally $mn$ such matrices of size $m \times n$

For $2 \times 3$ matrices:

$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$;

$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For column vectors $E_i$ in size $3$:

$E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Any vector

$X = \sum x_i E_i$

**Theorem 1.** If $A = (a_{ij})$ then

$A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}$. 

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 4
\end{pmatrix} = E_{11} + 2E_{12} + 3E_{13} + 4E_{23}
\]
Upper triangular matrices

\[ A = \sum_{i \leq j} a_{ij} E_{ij}, \]

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  0 & a_{22} & a_{23} & \cdots & a_{2n} \\
  0 & 0 & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix}
\]

Diagonal matrices

\[ D = \sum_{i} \lambda_i E_{ii}, \]

or

\[
D = \begin{pmatrix}
  \lambda_1 & 0 & 0 & \cdots & 0 \\
  0 & \lambda_2 & 0 & \cdots & 0 \\
  0 & 0 & \lambda_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & \lambda_n
\end{pmatrix}.
\]

\[ D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). \]

For example we have \( I = \text{diag}(1, 1, \ldots, 1) \) for the identity matrix.
**Definition 1.** A square matrix $P$ is **permutation matrix** if it has exactly one non-zero element in every row and every column and this element is equal to 1.

Example of a permutation matrix of size 4

$$P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}$$

If $s_i$ is the place of the 1 in row $i$ then all $s_i$ are different and

$$P = \sum_{i=1}^{n} E_{is_i}. \quad (1)$$

$$P = E_{12} + E_{24} + E_{33} + E_{41}$$

$$I = \sum_{i} E_{ii}$$

is a permutation matrix.
Multiplication by elementary and diagonal matrices. \( E_{ij}A = \).

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{pmatrix}
\]

\[
i \rightarrow \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
a_{j1} & a_{j2} & \cdots & a_{jj} & \cdots & a_{jn} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]

In row notation:

\[
E_{ij}A = E_{ij} \begin{pmatrix}
A_1 \\
\vdots \\
A_i \\
\vdots \\
A_m
\end{pmatrix} = \begin{pmatrix}
O \\
\vdots \\
A_j \\
\vdots \\
O
\end{pmatrix} \leftarrow i
\]
Multiplication from the right hand side works with the columns instead:

\[ AE_{kl} = (A_1 \ldots A_k \ldots A_l \ldots A_n)E_{kl} \]

\[ = (O \ldots O \ldots A_k \ldots O), \]

Column \( A_k \) moved to the place \( l \) and the other columns became zero.

\[ E_{ij}E_{kl} = \delta_{jk}E_{il}, \]

because the product is zero, if \( j \neq k \). Thus we have:

\[ AE_{kl} = \left( \sum_{i,j} a_{ij}E_{ij} \right)E_{kl} = \]

\[ = \sum_{i,j} a_{ij}E_{ij}E_{kl} = \sum_{i,j} a_{ij}\delta_{jk}E_{il} = \]

\[ = \sum_{i} a_{ik}E_{il}. \]
Multiplication with elementary square matrix

\[ I + cE_{ij} : \]

\[
(I + cE_{ij}) A = (I + cE_{ij}) \begin{pmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{pmatrix} =
\]

\[
\begin{pmatrix} A_1 \\ \vdots \\ A_i + cA_j \\ \vdots \\ A_m \end{pmatrix} \leftarrow i
\]

**Elementary row operation:**

Adding a multiple of the row \( j \) to the row \( i \). Naturally multiplying from the right hand side gives the corresponding effect on the columns.
**OBS!** The following theorem has no direct analogs for numbers.

**Theorem 2.** If $A$ is a nilpotent matrix then $I - A$ is invertible and $(I - A)^{-1} = I + B$ where $B$ is nilpotent.

**Proof.** If $A^k = 0$, then

$$I = I^k - A^k = (I - A)(I + A + A^2 + \cdots + A^{k-1})$$

and we note that the factors commute ($XY = YX$) since they are polynomials of the one and the same matrix $A$. 

$I + A$ is also invertible, because $-A$ is nilpotent too if $A$ is nilpotent.

**OBS!**

For $i \neq j$ the inverse of an elementary matrix:

$$(I + cE_{ij})^{-1} = I - cE_{ij}$$

because $E_{ij}^2 = 0$ for $i \neq j$. 

16
By this theorem, every triangular matrix that has ones on the main diagonal is invertible and has a triangular inverse with ones on the main diagonal. For example, by the proof of the theorem:

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{pmatrix}^{-1}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
- \begin{pmatrix}
0 & -2 & -3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
+ \begin{pmatrix}
0 & -2 & -3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & -2 & -3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}^2
= 
\begin{pmatrix}
1 & -2 & -3 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & -2 & -5 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]
Gauss Elimination and LU-decomposition

Echelon matrices. Pivot elements. Free and basic variables. Any system of linear equations can be written as a matrix equation

\[ AX = B. \]

\[
\begin{align*}
   x + 2y &= 3 \\
   4x + 5y &= 6 \\
   7x + 8y &= 9
\end{align*}
\]

\[ A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}. \]
If $A$ has the left inverse $V$ we can find the only possible candidate for a solution

$$VAX = VB \Rightarrow X = VB,$$

but $VB$ might be a false solution. **It is not clear if** $AVB = B$ If $V$ is also a right inverse, then $V = A^{-1}$ and $X = A^{-1}B$ is a solution.

Unfortunately this nice idea does not work when the inverse matrix does not exist.
the main idea = philosophy = strategy
Simplify the matrix without loosing the solutions and solve new system for the simplified matrix.

Echelon matrices, look as a staircase:

\[
\begin{pmatrix}
0 & * & * & * & * & * \\
0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

* arbitrary elements (even zeros)
The first * in rows are non-zero.
Echelon matrix of size \( m \times n \) looks as follows:
1) First \( r \) rows are nonzero,
2) The remaining \( m - r \) rows are zero.
3) If \( a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r} \) are the corresponding pivot elements then \( j_1 < j_2 < \cdots < j_r \).

\( r \) is called \textbf{rank} of the echelon matrix \( A \).
The columns \( j_1, j_2, \ldots, j_r \) are called \textbf{pivot columns}.
For the system of equations \( AX = B \)
\( x_{j_1}, x_{j_2}, \ldots, x_{j_r} \) are called \textbf{basic variables}.
The remaining variables are called \textbf{free variables}.
\[ A = \begin{pmatrix}
0 & 2 & 0 & 0 & 7 & 4 & 4 \\
0 & 0 & 3 & 0 & 0 & 6 & 9 \\
0 & 0 & 0 & 0 & 0 & 2 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \]

The rank is equal to 3
pivot elements are underlined, and so 2, 3, 6
are pivot columns;
\( x_2, x_3 \) and \( x_6 \) are basic variables
\( x_1, x_4, x_5 \) and \( x_7 \) are free variables.
$AX = B$ has the following solutions.
If $b_4 \neq 0$ or $b_5 \neq 0$, there are no solutions.
If $b_4 = b_5 = 0$ we choose parameters for free variables, say $x_1 = t, x_4 = s, x_5 = u, x_7 = v$ and calculate the basic variables starting from the last non-zero equation:

$$2x_6 + 8x_7 = b_3 \Rightarrow 2x_6 + 8v = b_3 \Rightarrow x_6 = -4v + \frac{b_3}{2}.$$ 

Now we can substitute this into second but last non-zero equation:

$$3x_3 + 6x_6 + 9x_7 = b_2 \Rightarrow$$

$$3x_3 + 6 \left( -4v + \frac{b_3}{2} \right) + 9v = b_2 \Rightarrow$$

$$x_3 = 5v - b_3 + \frac{b_2}{3}.$$ 

And at last we come to the first equation:

$$2x_2 + 7x_5 + 4x_6 + 4x_7 = b_1 \Rightarrow$$

$$2x_2 + 7u + 4 \left( -4v + \frac{b_3}{2} \right) + 4v = b_1 \Rightarrow$$

$$x_2 = -\frac{7u}{2} + 6v - b_3 + \frac{b_1}{2}.$$
Gauss elimination. LU-decomposition

How to transform an arbitrary matrix \( A \) to echelon form without losing possible solutions?

1) If all rows are zero it is already on echelon form.
2) If not, find the first non-zero column \( j \) and consider its first non-zero element \( a \).
3) If it is not on the first place, exchange its row with the first row.
4) Now \( a \) is the pivot element of the first row \( R \). Use it to "clear" all other elements in the column \( j \), using elementary row operations. More exactly:
   if \( a_{ij} \neq 0 \) then replace the row \( R_i \) by \( R_i - \frac{a_{ij}}{a} R \).
5) When all is done forget the first row and repeat the procedure with the remaining rows.

OBS

Of course, the first of the remaining rows will be not the first in the original matrix, but if we forget it we can repeat the process. We need only to write the correct indices - that is why we have written \( R \) instead for \( R_1 \).
**Very important!**
We can do Gauss elimination using multiplications by elementary matrices and permutations.

\[ A = \begin{pmatrix}
0 & 0 & 1 & 2 & 3 \\
0 & 2 & 4 & 6 & 8 \\
0 & 4 & 8 & 12 & 16 \\
0 & 1 & 2 & 3 & 5 \\
\end{pmatrix}. \]

The first non-zero column is second one and it has a non-zero element on the second place. So first we need to exchange the first and second rows.
We do it using multiplication on the left by a permutation matrix \( P_1 \):

\[
P_1A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 2 & 3 \\
0 & 2 & 4 & 6 & 8 \\
0 & 4 & 8 & 12 & 16 \\
0 & 1 & 2 & 3 & 5 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 4 & 8 & 12 & 16 \\
0 & 1 & 2 & 3 & 5
\end{pmatrix}.
\]

We get pivot element 2 in the first row and now use it to clean the column.
We start from the third row, because the second one is already clean. We need to subtract twice first row to "kill" 4 in the third row. The corresponding elementary matrix is

\[ L_1 = I - 2E_{31}. \]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 4 & 8 & 12 & 16 \\
0 & 1 & 2 & 3 & 5
\end{pmatrix}
= \\
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 5
\end{pmatrix}.\]
Now it is the turn of the fourth row. To "kill" 1 we need to subtract \( \frac{1}{2} \) of the first row. From the multiplication point of view we need the elementary matrix \( L_2 = I - \frac{1}{2}E_{41} \):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{1}{2} & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The column is clear and we can forget about the first row, working in the same style with the remaining rows, i.e. with the matrix

\[
\begin{pmatrix}
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
It is more practical to write the whole matrix ignoring the first row. We see that next non-empty column we need to work with is the third one (not the second, because 2 is in the ignored first row). We have here the non-zero element 1 already in the very beginning (4 is ignored) so we get a new pivot element:

$$
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The rest of the column is already clean, so we are done here and can forget the second row too.
In the remaining two rows we need only one small permutation $P_2$ and the echelon form is obtained:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
If we denote the obtained echelon matrix $U$ we can conclude that

$$P_2L_2L_1P_1A = U.$$  

This can be slightly improved.  

**Theorem 3.** *For every matrix $A$ there exist a permutation matrix $P$, an echelon matrix $U$ and a lower triangular matrix $L$ with ones on the main diagonal such that*

$$PA = LU.$$  

*This is called the (or $LU$-factorization) of $A$.***
Proof We can perform Gauss elimination to get an echelon form $U$ using the multiplications with the elementary matrices $I + cE_{ij}$ with $i > j$ and permutation matrices $P_i$. But we can slightly change the order. First we collect all the permutations in one – call it $P$ and only after that perform the multiplications with the elementary matrices. Indices in those probably will change, but they will still be elementary matrices $I + cE_{ij}$ with $i > j$. The result is

$$L_k L_{k-1} \cdots L_1 PA = U$$

This gives

$$PA = L_1^{-1} \cdots L_k^{-1} U = LU.$$ 

It remains to note that the inverse of any elementary matrix is an elementary matrix of the same form:

$$(I + cE_{ij})^{-1} = I - cE_{ij}$$

if $i > j$. So $L$ as a product of lower triangular matrices is itself triangular and has ones on the main diagonal as them (prove this as a good exercise).
Let us check how this works in our example. The complete permutation is \( P = P_2P_1 \) and we have

\[
P A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 & 2 & 3 \\
0 & 2 & 4 & 6 & 8 \\
0 & 4 & 8 & 12 & 16 \\
0 & 1 & 2 & 3 & 5
\end{pmatrix} =
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 5 \\
0 & 4 & 8 & 12 & 16
\end{pmatrix}
\]

\( L_1 = I - \frac{1}{2}E_{31} \) is not the same (but it reminds us of the old \( L_2 \)).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\frac{1}{2} & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 5 \\
0 & 4 & 8 & 12 & 16
\end{pmatrix} =
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 4 & 8 & 12 & 16
\end{pmatrix}
\]
With $L_2 = I - 2E_{41}$ we get

$$U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-2 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 4 & 8 & 12 & 16
\end{pmatrix} =
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
So we know $P, U$ and need only to calculate $L$:

$$L = L_1^{-1} L_2^{-1} = \left( I - \frac{1}{2} E_{31} \right)^{-1} (I - 2E_{41})^{-1} =$$

$$\left( I + \frac{1}{2} E_{31} \right) (I + 2E_{41}) =$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
2 & 0 & 0 & 1
\end{pmatrix}$$

$$= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & 1 & 0 \\
2 & 0 & 0 & 1
\end{pmatrix}.\]
Definition. The rank \( r(A) \) of a matrix \( A \) is the rank of the matrix \( U \) in its LU-decomposition: \( PA = LU \), i.e. the number \( r \) of pivot elements in \( U \).

A practical way to find the rank of a matrix is to make the Gauss elimination and calculate the number of non-zero rows in the obtained matrix \( U \).

For example the matrix \( A \) in the previous example has rank 3.