Öresund Non-commutative analysis and symmetry course

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Copenhagen, Lund

OH slides
Lecture 1. + .... extra
Non-commutative analysis

Non-commutative geometry and algebra
Operator algebras
Representation Theory
Lie analysis, Lie algebras
Symmetry analysis

Classical and quantum physics,
Some engineering subjects such as
signal processing, control theory

Algebraic operator analysis
of differential, integral, difference-like equations

Dynamical systems,
Actions of groups and semigroups,
wavelets
\begin{align*}
AB &= BA \\
AB &= -BA \\
AB &= q \cdot BA^d \\
\frac{i}{\hbar} (AB - BA) &= I \\
AB + BA &= I \\
A^2 + B^2 &= I \\
P(A, B) &= 0 \\
AB &= B(\alpha I + c A)(\beta I + dA)^{-1} \\
A_i A_j - A_j A_i &= \sum_{k=1}^{n} C_{i,j}^k A_k \\
A_i A_j - E_{ij} A_i A_j &= \sum_{k=1}^{n} C_{i,j}^k A_k \\
AB &= BF(A)
\end{align*}
Important non-commutative algebras
and non-commutative matrix equations

\[ AX = XB \]
\[ AX - XB = C \]
\[ AX - XF(A) = C \]
\[ F(A)X - XA = C \]
\[ AX = XF(A) \]
\[ F(A)X = XA \]

\[ X^{-1}AX = B \]

Examples:
\[ PQ - QP = I \]
\[ PQ + QP = I \]
\[ PQ - QP = I \]

\[ X^{-1}AX = F(A) \]
\[ XAX^{-1} = F(A) \]

OBS!
All these commutation relations (equations) are linear in X

General class of semi-linear commutation relations (equations)

\[ \sum_{j=0}^{n} A_j X B_j = C \]
\[ \sum_{j=0}^{n} A_j X F(A_j) = C \]
\[ \sum_{j=0}^{n} F(A_j) X A_j = C \]
Lie algebras and their deformations and generalizations

\[ A_j A_k - A_k A_j = \sum_e C^e_{jk} A_e \]

\[ A \rightarrow e^A \]

Lie groups, Continuous transformation groups

One of the main applications: Symmetries of equations

\[ T x = f \Rightarrow T(S x) = f \]

\( x \) is solution \( \Rightarrow S x \) is solution

\( S \) is a symmetry of equation

\( S_1, S_2 \) are symmetries \( \Rightarrow S_1 S_2 \) is symmetry

Deformations, generalizations, ...

\[ \mu_{jk} A_j A_k - g_{kj} A_k A_j = \sum_e C^e_{jk} A_e \]

\[ \mu_{jk} A_j A_k - g_{kj} A_k A_j = F_{jk}(A_1, A_2, \ldots) \]
Quantum plane
commutation relation

\[ AB = q BA \]

Rotation C*-algebra

\[ UV = q VU, \quad \|q\| = 1 \]

\[ u^* = u^{-1}, \quad v^* = v^{-1} \]

Weyl form of Heisenberg Canonical Commutation relations of Quantum Mechanics

\[ u_t \cdot v_s = e^{i ts} v_s u_t \]

\[ AB = q BA = B(qA) \]

\[ AB = BF(A), \quad F: C \rightarrow C \]

Cross product algebra

Covariance relation

\[ B^{-1} AB = F(A) \]
\[ pQ - Qp = 1 \]
\[ pQ^n - Q^n p = n Q^{n-1} = (Q^n)' \]
\[ p f(q) - f(q) p = f'(q) \]
\[ f(q) = e^{sQ}, \quad f'(q) = se^{sQ} \]
\[ pe^{sQ} - e^{sQ} p = se^{sQ} \]
\[ (p - s1) e^{sQ} = e^{sQ} p \quad \text{pe}^{sQ} = e^{sQ} (p + s1) \]
\[ (p - s1)^n e^{sQ} = e^{sQ} p^n \]
\[ g(p - s1) e^{sQ} = e^{sQ} g(p) \]
\[ g(x) = e^{tx} \]
\[ e^{t(p - s1)} e^{sQ} = e^{sQ} e^{tp} \]
\[ e^{tp} e^{ts} e^{sQ} = e^{sQ} e^{tp} \]
\[ e^{tp} e^{sQ} = e^{ts} e^{sQ} e^{tp} \]
\[ e^{tp} e^{sQ} = e^{ts} e^{sQ} e^{tp} \]
\[ ut vs = e^{ts} vs ut \]
\[ A B = q B A \]

Weyl form of Heisenberg relations
Solution of equations by Operational Methods and "dynamical" commutation relations

\[ AB - BA = I \]

\[ \downarrow \]

\[ Ae^{\lambda B} = e^{\lambda B} (A + \lambda I) \]

\[ (A - \lambda I)e^{\lambda B} = e^{\lambda B}A \]

\[ A - \lambda I = (A - \lambda I)e^{\lambda B}e^{-\lambda B} = e^{\lambda B}Ae^{-\lambda B} \]

If \( A \) has right inverse \( A_{r}^{-1} \), \( AA_{r}^{-1} = I \)

\[ \frac{1}{A}y = A_{r}^{-1}y + \text{Ker } A, \quad \text{Ker } A = \{ h \mid Ah = 0 \} \]

\[ \frac{1}{A - \lambda I} = e^{\lambda B} \frac{1}{A} e^{-\lambda B} \]

General solution of \((A - \lambda I)y = f\)

\[ y = \frac{1}{A - \lambda I}(f) = e^{\lambda B} \frac{1}{A} e^{-\lambda B}(f) \]

General solution of \( \sum_{k=0}^{n} a_{k}A^{k}y = f \)

\[ \sum_{k=0}^{n} a_{k}A^{k} = a_{n} \prod_{i=1}^{n}(A - \lambda_{i}I) = \]

\[ = a_{n} e^{\lambda_{1}B}A e^{-\lambda_{1}B} \ldots e^{\lambda_{n}B}A e^{-\lambda_{n}B} \]

\[ = a_{n} e^{\lambda_{1}B}A e^{-\lambda_{1}B} \ldots e^{\lambda_{n}B}A e^{-\lambda_{n}B} \]

\[ y = e^{\lambda_{1}B} \frac{1}{A}e^{-\lambda_{1}B} \ldots e^{\lambda_{n}B} \frac{1}{A}e^{-\lambda_{n}B} \]

\[ \frac{\partial n}{\partial n} = \frac{1}{P(A)f} \]
Heisenberg algebra

\[ PQ - QP = 1 \]

\[ \text{dim} H < \infty \text{ Matrix representation?} \]
\[ \text{dim} H \neq 0 \text{ representation?} \]

\[ \text{tr}(PQ - QP) = \text{dim} H \]
\[ \text{tr}(PQ) - \text{tr}(QP) = 0 \]

!!! NO finite-dimensional matrix representations

\[ \sum_{ij} a_{ij} \overline{p}_i p_j \]

Winter, Wildand

Theorem: 1) No bounded operators;
2) No elements in any unital normed algebra

\[ \text{Spec}(x) = \{ \lambda \in \mathbb{C} | x - \lambda I \text{ has no } (x-\lambda I) \} \]

\[ \text{Spec}(a \ b) \cup \{ 0 \} = \text{Spec}(a \ b) \cup \{ 0 \} \]

\[ \text{Spec} \left( \sum_{ij} a_{ij} \overline{p}_i p_j \right) \]

\[ \text{Spec}(PQ) = \text{Spec}(1 + PQ) = 1 + \text{Spec}(PQ) \]

Shift dynamical system: \( \lambda \mapsto 1 + \lambda \)

\[ \text{Spec}(PQ) \text{ unbounded set } \Rightarrow \text{ PQ unbounded} \]
Matrix and Operator representations of rotation algebra

\[ AB = g \cdot BA \ , \ g \neq 1 \]
\[ AB = B( gA) = BF(A) , \ F: \lambda \rightarrow g \lambda \]
\[ A: H \rightarrow H , \ B: H \rightarrow H \]

I. Rational rotation algebra, \( g^n = 1 \)
\[ \lambda \rightarrow q \lambda \rightarrow q^2 \lambda \rightarrow \cdots \rightarrow q^{n-1} \lambda \rightarrow q^n \lambda = \lambda \]

\[ A_\lambda = \begin{pmatrix} 0 & q^\lambda & 0 & \cdots & 0 \\ 0 & 0 & q^{2 \lambda} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^{(n-1) \lambda} \end{pmatrix} , \ B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \]

\[ A_\lambda B \begin{pmatrix} \frac{2 \lambda}{n} \\ \vdots \\ \frac{2 \lambda}{n} \end{pmatrix} = A_\lambda \begin{pmatrix} e^{i \theta} & \cdots & e^{i \theta} \\ \vdots & \ddots & \vdots \\ e^{i \theta} & \cdots & e^{i \theta} \end{pmatrix} = 
\begin{pmatrix} e^{i \theta} q^{2 \lambda} & \cdots & e^{i \theta} q^{(n-1) \lambda} \\ \vdots & \ddots & \vdots \\ e^{i \theta} q^{(n-1) \lambda} & \cdots & e^{i \theta} \end{pmatrix} = g^{-1} A_\lambda B \begin{pmatrix} \frac{2 \lambda}{n} \\ \vdots \\ \frac{2 \lambda}{n} \end{pmatrix} \]

\[ BA_\lambda \begin{pmatrix} \frac{2 \lambda}{n} \\ \vdots \\ \frac{2 \lambda}{n} \end{pmatrix} = \begin{pmatrix} e^{i \theta} q^{-1} & \cdots & e^{i \theta} q^{n-1} \\ \vdots & \ddots & \vdots \\ e^{i \theta} q^{n-1} & \cdots & e^{i \theta} \end{pmatrix} = g^{-1} A_\lambda B \begin{pmatrix} \frac{2 \lambda}{n} \\ \vdots \\ \frac{2 \lambda}{n} \end{pmatrix} \]

\[ AB = g \cdot BA \iff g^n = 1 \]

Periodic rotation, \( \lambda \rightarrow g \lambda \)
\[ q = e^{i \frac{2 \pi \lambda}{n}} \]

Obs!
\[ A = \begin{pmatrix} 2 & 0 & \ldots & 0 \\ 0 & 2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \]

Properties of \( A_\lambda, B, \lambda \neq 0 \)

1. \( A_\lambda^* A_\lambda = A_\lambda A_\lambda^* \), \( A_\lambda \) is a normal matrix (operator)

2. \( B B^* = B^* B = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( B \) is unitary

3. No proper invariant subspace for \( A_\lambda, B, A_\lambda^*, B^* \)

\( H_0 \subset H \) no such \( H_0 \)

\( H_0 \neq \{0\}, H_0 \neq H \)

\( A_\lambda(H_0) \leq H_0, B(H_0) \leq H_0 \)

Th. Any \( A, B \) such that \( AB = qBA \)

1. \( A \) is normal, 2. \( B \) is unitary

\[ A = \begin{pmatrix} A_\lambda & 0 & \cdots & 0 \\ 0 & A_\lambda & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A_\lambda \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{pmatrix} \]

in some orthonormal basis

\( \leftrightarrow \) up to unitary equivalence

\( \tilde{A} = U^{-1} A U, \tilde{B} = U^{-1} B U \)

\( U \) is unitary matrix of orthonormal basis change
What about other solutions to $AB = qBA$?

**Obs!** Very interesting question!!!

**Example:** $\dim H = n$

$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1n} \\ 0 & w_{22} & w_{23} & \cdots & w_{2n} \\ 0 & 0 & w_{33} & \cdots & w_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & w_{nn} \end{pmatrix}$

$AB = \begin{pmatrix} 0 & qw_{12} & qw_{13} & \cdots & qw_{1n} \\ 0 & q^2 w_{22} & q^2 w_{23} & \cdots & q^2 w_{2n} \\ 0 & 0 & q^3 w_{33} & \cdots & q^3 w_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q^{n-1} w_{nn} \end{pmatrix}$

$BA = \begin{pmatrix} 0 & w_{11} & w_{12} & w_{13} & \cdots & w_{1n-1} \\ 0 & 0 & w_{22} & w_{23} & \cdots & w_{2n-1} \\ 0 & 0 & 0 & w_{33} & \cdots & w_{3n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & w_{nn-1} \end{pmatrix}$

$AB = qBA$ **Claim !!!**

Everything works because $0$ is a fixed point of the mapping (dynamical system)

$\lambda \mapsto q\lambda$

$0 \mapsto q\cdot 0 = 0$
Operator representations of
$A B = q B A$

quantum plane

Example:
$\dim H = \infty$

$A = \begin{pmatrix}
  1 & 0 \\
  0 & q^2 \\
  0 & 0 & q^2 \\
  0 & 0 & 0 & q^4
\end{pmatrix},
B = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}$

basis: $\bar{e}_1, \bar{e}_2, \bar{e}_3, \ldots$

$A B \bar{e}_k = A \bar{e}_{k+4} = q^k \bar{e}_{k+4} = q B A$
$B A \bar{e}_k = B (q^{k-4} \bar{e}_k) = q^{k-4} \bar{B} \bar{e}_k = q^{k-4} \bar{e}_{k+4}$

Example: Rescaling operator !!!

Very important in physics !!!
and other applications

$A = T_q f(x) \mapsto f(qx)$
$B = M_x f(x) \mapsto x f(x)$
$A B f(x) = q x f(qx) = q B A f(x)$