INTERPOLATION SPACES AND INTERPOLATION GROUPS

This is a temporary version of the notes which were used for the second part of my lecture of 30/9/2010. Hopefully a somewhat more definitive version will be available later.

Some parts of the lecture do not appear in these notes, and of course we did not get to the latter part of these notes in the lecture.

1. Compatible couples of quasinormed abelian groups.

Definition 1. Let A be an abelian group where the group operation is denoted by $+$ and the identity element by 0. A functional $\|\cdot\|$ on A is said to be a quasinorm or a c-quasinorm on A if it satisfies

(i) $\|a\| \geq 0$ for all $a \in A$.

(ii) $\|a\| = 0$ if and only if $a = 0$.

(iii) $\|-a\| = \|a\|$ for all $a \in A$.

(iv) There exists a positive constant c such that $\|a+b\| \leq c(\|a\| + \|b\|)$ for all a and b in A.

A quasinormed abelian group is thus an abelian group equipped with a quasinorm. If $c = 1$ then the 1-quasinorm $\|\cdot\|$ is also referred to as a norm, and we have a normed abelian group.

Theorem 2. *Let A be a c-quasinormed abelian group with c-quasinorm denoted by $\|\cdot\|_A$. Let ρ be the positive number which satisfies $(2c)^\rho = 2$. Then there exists a 1-quasinorm $\|\cdot\|_A^*$ on A such that*

$$\|a\|_A^* \leq \|a\|_A^\rho \leq 2\|a\|_A^* \quad \text{for all } a \in A\,. \tag{1.1}$$

This important and useful result, which is apparently due to Aoki and Rolewicz (independently??), appears as Lemma 3.10.1 on pp. 59–60 of [2] and the proof will be taken from there. However it might be convenient to present one part of the proof in [2] more explicitly as the following separate lemma.

Lemma 3. *Suppose that $n \geq 2$ and that ν_1, ν_2,.....,ν_n are non negative integers such that*

$$\sum_{k=1}^{n} 2^{-\nu_k} \leq 1\,. \tag{1.2}$$

Then there exists a non empty subset I_1 of $\{1,2,...,n\}$ such that $I_2 = \{1,2,...,n\}\backslash I_1$ is also non empty and

$$\sum_{k \in I_1} 2^{-\nu_k} \leq \frac{1}{2} \text{ and } \sum_{k \in I_2} 2^{-\nu_k} \leq \frac{1}{2}\,.$$

Proof. If $\sum_{k=1}^{n} 2^{-\nu_k} \leq 1/2$ then every non empty subset of $\{1,2,...,n\}$ which is not all of $\{1,2,...,n\}$ will have the required property. So let us suppose that $\sum_{k=1}^{n} 2^{-\nu_k} > 1/2$. We may also suppose without loss of generality, that

$$\nu_1 \leq \nu_2 \leq .... \leq \nu_n\,. \tag{1.3}$$

Let m be the largest integer such that $\sum_{k=1}^{m} 2^{-\nu_k} \leq \frac{1}{2}$. Since the condition (1.2) implies that $\nu_1 \geq 1$, we see that $1 \leq m < n$. We will show that in fact

$$\sum_{k=1}^{m} 2^{-\nu_k} = \frac{1}{2} \tag{1.4}$$

and this will ensure that the set $I_1 = \{1,2,...,m\}$ has the required property. The condition (1.3) ensures that $\sum_{k=1}^{m} 2^{-\nu_k}$ can be written in the form $p \cdot 2^{-\nu_m}$ for some integer p. Then the maximality of m and another application of (1.3) imply that $p \cdot 2^{-\nu_m} \leq 2^{-1} < (p+1) \cdot 2^{-\nu_m}$. This implies, in turn, that $p \leq 2^{\nu_m - 1} < p+1$. Since $2^{\nu_m - 1}$ is an integer, it must equal p. This means that (1.4) holds, and completes the proof of the lemma. $\square$

In view of Theorem 2, each quasinormed abelian group A has a metric given by $d(a,a') = \|a-a'\|_A^*$. We can define the notion of a Cauchy sequence in terms of this metric, or equivalently, in terms of the original quasinorm. Similarly A will be complete with respect to this metric if and only if it is complete with respect to the original quasinorm. In general (in contrast to the earlier versions of abstract interpolation theory) we will not require our quasinormed abelian groups to be complete. However it will sometimes be convenient (see Theorem 10) to consider the case where they are complete.

It will be rather convenient if we can define and work with the K-functional, J-functional and the "spaces" or groups $(A_0, A_1)_{\theta,q}$ in the case of a couple (A_0, A_1) where A_0 and A_1 are merely quasinormed abelian groups. The theory in this case has been worked out in detail in [10] and also summarized in [2] pp. 59–69.

We need to impose some minimal compatibility conditions on A_0 and A_1. (See [10] p. 225.) These will be appear to be somewhat less demanding than conditions which are normally imposed in discussions of the case where A_0 and A_1 are Banach spaces or quasinormed spaces, but in fact they are essentially equivalent to those conditions. The only significant difference is that we have simply stopped thinking about or even requiring the operation of multiplication by scalars.

We will require the quasinormed abelian groups A_0 and A_1 to both be subgroups of some larger abelian group $\mathcal{A}$. This means that, for $j = 0,1$ the group operation on A_j is the restriction to A_j of the group operation (to be denoted by $+$) on $\mathcal{A}$. It also ensures that $A_0 \cap A_1$ and $A_0 + A_1$ are both abelian groups, with respect to the same operation $+$. We want these two groups to be quasinormed, respectively, by the two functionals $J(t,a;A_0,A_1)$ and $K(t,a;A_0,A_1)$ for each fixed positive t, which we will now define. Suppose that A_j is c_j-quasinormed for $j = 0,1$. Then it is clear that the functional

$$J(t,a;A_0,A_1) := \max\left\{\|a\|_{A_0}, t\|a\|_{A_1}\right\} \text{ for all } a \in A_0 \cap A_1$$

defines a c-quasinorm on $A_0 \cap A_1$ for $c = \max\{c_0, c_1\}$. It is also clear that the functional on $A_0 + A_1$ defined by

$$K(t,a;A_0,A_1) := \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\right\}$$

satisfies conditions (i), (iii) and (iv) of Definition 1, again with $c = \max\{c_0, c_1\}$. However condition (ii) may fail to hold. To ensure that it does hold we need an additional condition. In the usual discussion of interpolation theory, when A_0 and A_1 are assumed to be normed linear spaces or Banach spaces, it is customary to require that A_0 and A_1 are both continuously embedded in a Hausdorff topological linear space. (This can in fact be shown to be equivalent to requiring A_0 and A_1 to both be continuously embedded in a normed linear space.) Here, in the more general context of quasinormed abelian groups, we could analogously require A_0 and A_1 to be continuously embedded in a Hausdorff topological group or simply in a Hausdorff topological space, or in a quasinormed or normed abelian group. Instead we will impose a different "compatibility of convergence" condition, as is done in the paper [10].

(*) *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of elements in* $A_0 \cap A_1$ *and let* $a \in A_0$ *and* $b \in A_1$ *be such that* $\lim_{n\to\infty}\|a_n - a\|_{A_0} = 0$ *and* $\lim_{n\to\infty}\|a_n - b\|_{A_1} = 0$. *Then* $a = b$.

In [10] this condition is referred to as a "separability axiom". At first sight it seems to be weaker than other possible conditions mentioned above that might be imposed. But (see Remark 6 below) it turns out to be essentially equivalent to them.

As in [10], we will use the terminology *quasinormed abelian couple* to mean a couple (A_0, A_1) of quasinormed abelian groups A_0 and A_1 which are both subgroups of some other abelian group and which satisfy the condition (*).

Remark 4. There must surely be examples of abelian groups $\mathcal{A}$ which contain two quasinormed abelian groups A_0 and A_1 but for which the condition (*) does not hold. At this moment I cannot offhand think of such an example. My feeling is that at this stage we have more important and interesting things to do than look for such an example. But that is a matter of personal taste.

Exercise 5. Show that Condition (*) indeed suffices to ensure that the K-functional satisfies condition (ii) of Definition 1.

Remark 6. In particular the result of Exercise 5 tells us that $A_0 + A_1$ is a quasinormed abelian group and that its quasinorm satisfies $\|a\|_{A_0+A_1} \le \|a\|_{A_j}$ for all $a \in A_j$ for $j = 0,1$. So, after all, we see that condition (*) is *equivalent* to a sort of topological requirement on $\mathcal{A}$, namely that:

(**) *A_0 and A_1 are both subgroups of a quasinormed abelian group $\mathcal{A}$ whose quasinorm $\|\cdot\|_{\mathcal{A}}$ satisfies $\|a\|_{\mathcal{A}} \le \alpha_j \|a\|_{A_j}^{\beta_j}$ for all $a \in A_j$ and for $j = 0, 1$, where α_j and β_j are positive constants.*

This is because, after we are given that condition (*) holds, we have the option of changing our original choice of the group $\mathcal{A}$ which contains A_0 and A_1 and instead choosing $\mathcal{A}$ to be the (possibly smaller) group $A_0 + A_1$ and choosing $\beta_0 = \beta_1 = 1$. The reverse implication (**)$\Rightarrow$(*) is obvious.

Definition 7. Let (A_0, A_1) be a quasinormed abelian couple. For each $\theta \in (0, 1)$ and each $q \in (0, \infty]$ let $(A_0, A_1)_{\theta,q}$ be the set of all elements $a \in A_0 + A_1$ for which the quantity

$$\|a\|_{(A_0,A_1)_{\theta,q}} := \left(\int_0^\infty \left(t^{-\theta} K(t, a; A_0, A_1) \right)^p \frac{dt}{t} \right)^{1/p}$$

is finite. When $q = \infty$, the definition is modified in the usual way, and we have

$$\|a\|_{(A_0,A_1)_{\theta,\infty}} := \sup_{t>0} t^{-\theta} K(t, a; A_0, A_1) \,.$$

Exercise 8. Check that $\|\cdot\|_{(A_0,A_1)_{\theta,q}}$ is a quasinorm and that $(A_0, A_1)_{\theta,q}$ is a quasinormed abelian group for each $\theta \in (0, 1)$ and each $q \in (0, \infty]$.

Exercise 9. Obviously $K(t, a; A_0, A_1)$ is a non decreasing function of t for each fixed $a \in A_0 + A_1$. Find an exact formula connecting $K(t, a; A_1, A_0)$ and $K(1/t, a; A_0, A_1)$ for each $t > 0$ and use it to show that $\frac{1}{t} K(t, a; A_0, A_1)$ is a non increasing function of t. Use the same formula to also show that $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$ isometrically for each $\theta \in (0, 1)$ and each $q \in (0, \infty]$.

Theorem 10. *Let (A_0, A_1) be a compatible couple of quasinormed abelian groups satisfying the condition (*). If both A_0 and A_1 are complete, then $(A_0, A_1)_{\theta,q}$ is also complete for each $\theta \in (0, 1)$ and each $q \in (0, \infty]$.*

Proof. We refer to pp. 239–240 of [10]. $\square$

2. Some other equivalent definitions of the quasinormed groups $(A_0, A_1)_{\theta,q}$

In our first equivalent definition we see that the continuous variable $t \in (0, \infty)$ can be replaced by the discrete variable $n \in \mathbb{Z}$

Theorem 11. *For each positive constant $\lambda \neq 1$ the quasinorms*

$$\left(\sum_{n\in\mathbb{Z}} \left(\lambda^{-n\theta} K(\lambda^n, a; A_0, A_1) \right)^q \right)^{1/q} \quad \text{and} \quad \|a\|_{(A_0,A_1)_{\theta,q}}$$

are equivalent for each $\theta \in (0, 1)$ and each $q \in (0, \infty]$ and the constants of equivalence depend only on λ, θ, and q.

"Proof". I will leave the proof of this theorem as an exercise. The underlying idea is that the functions $t \mapsto t^{-\theta}$ and $t \mapsto K(t, a; A_0, A_1)$ are both equivalent to contant functions on each interval of the form $[\lambda^n, \lambda^{n+1}]$. In particular, the relevant constants of equivalence do not depend on n.

Corollary 12. *Let (A_0, A_1) be a quasinormed abelian couple. Then, for each $\theta \in (0, 1)$ and each q_1 and q_2 satisfying $0 < q_1 < q_2 \le \infty$ we have $(A_0, A_1)_{\theta,q_1} \subset (A_0, A_1)_{\theta,q_2}$ and the inequality $\|a\|_{(A_0,A_1)_{\theta,q_2}} \le C \|a\|_{(A_0,A_1)_{\theta,q_1}}$ holds for all $a \in (A_0, A_1)_{\theta,q_1}$ and for a constant C which depends only on θ, q_1 and q_2.*

Proof. We simply apply Theorem 11 with, for example $\lambda = 2$, together with the fact that $\ell^{q_1} \subset \ell^{q_2}$ with $\left\|\{\alpha_n\}_{n\in\mathbb{Z}}\right\|_{\ell^{q_2}} \le \left\|\{\alpha_n\}_{n\in\mathbb{Z}}\right\|_{\ell^{q_1}}$ for every sequence $\{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^{q_1}$. (This last inequality is obvious in the special case where $q_2 = \infty$, and the general case can be readily deduced from that special case.) $\square$

Next we describe a seemingly quite different construction of interpolation groups. It uses the J-functional instead of the K-functional. As in Theorem 11, we only have to consider a discrete countable set of values of t of the form $t = \lambda^n$ for some constant $\lambda > 1$.

Definition 13. For each fixed $\theta \in (0,1)$, $q \in (0,\infty]$ and positive λ with $\lambda \neq 1$, the set $(A_0,A_1)_{\theta,q,J,\lambda}$ consists of all elements $a \in A_0 + A_1$ with the following property:

There exists a sequence $\{u_n\}_{n\in\mathbb{Z}}$ of elements of $A_0 \cap A_1$ such that

$$\lim_{N\to\infty} \left\| \sum_{n=-N}^{N} u_n - a \right\|_{A_0+A_1} = 0 \tag{2.1}$$

and

$$\left(\sum_{n\in\mathbb{Z}} \left(\lambda^{-n\theta} J(\lambda^n, u_n; A_0, A_1) \right)^q \right)^{1/q} < \infty \,, \tag{2.2}$$

The functional $\|a\|_{(A_0,A_1)_{\theta,q,J,\lambda}}$ is defined for each element $a \in (A_0,A_1)_{\theta,q,J,\lambda}$ by the formula

$$\|a\|_{(A_0,A_1)_{\theta,q,J,\lambda}} = \inf \left(\sum_{n\in\mathbb{Z}} \left(\lambda^{-n\theta} J(\lambda^n, u_n; A_0, A_1) \right)^q \right)^{1/q}$$

where the infimum is taken over all $A_0 \cap A_1$ valued sequences $\{u_n\}_{n\in\mathbb{Z}}$ satisfying (2.1) and (2.2).

In fact this set is the same as $(A_0,A_1)_{\theta,q}$.

We will refer to the functional $\|\cdot\|_{(A_0,A_1)_{\theta,q,J,\lambda}}$ as the quasinorm of $(A_0,A_1)_{\theta,q,J,\lambda}$. This is a temporary abuse of language because we still have to prove that it really is a quasinorm. The following exercise will contribute to doing that.

Exercise 14. Show that the functional $\|\cdot\|_{(A_0,A_1)_{\theta,q,J,\lambda}}$ has the properties (i), (iii) and (iv) of Definition 1. (Show that (iv) holds for a constant c which is sufficiently large so that both A_0 and A_1 are c-quasinormed. The remaining property (ii) of Definition 1 will follow immediately from Theorem 16 and the fact, which we already know, that $\|\cdot\|_{(A_0,A_1)_{\theta,q}}$ is a quasinorm.)

Exercise 15. Show that $(A_0,A_1)_{\theta,q,J,\lambda} = (A_0,A_1)_{\theta,q,J,1/\lambda}$ and that the quasinorms of these two groups are equal.

Theorem 16. *Let (A_0,A_1) be a quasinormed abelian couple and that A_0 is c_0-quasinormed and A_1 is c_1-quasinormed. Then, for each $\theta \in (0,1)$, each $\lambda > 1$ and each $q \in (0,\infty]$, the sets $(A_0,A_1)_{\theta,q,J,\lambda}$ and $(A_0,A_1)_{\theta,q}$ coincide and the functionals $\|a\|_{(A_0,A_1)_{\theta,q}}$ and $\|a\|_{(A_0,A_1)_{\theta,q,J,\lambda}}$ are equivalent. The constants of equivalence depend only on θ, λ, q, c_0 and c_1.*

We will first present one component of the proof of Theorem 16 as a separate lemma. Over the years various versions of this lemma have been given the perhaps slightly too pompous title of "the fundamental lemma of interpolation theory". In this lemma, and indeed in the rest of the proof of Theorem 16, we will not worry very much about the size of the constants.

Lemma 17. *Let (A_0,A_1) be a quasinormed abelian couple. Suppose that A_0 is c_0-quasinormed and A_1 is c_1-quasinormed, and let $c = \max\{c_0, c_1\}$. Let $\lambda > 1$ be a constant.*

Let a be an element of $A_0 + A_1$ which satisfies the two conditions

$$\lim_{t\to 0} K(t,a;A_0,A_1) = 0 \ \text{ and } \ \lim_{t\to\infty} \frac{1}{t} K(t,a;A_0,A_1) = 0 \,. \tag{2.3}$$

Then there exists a sequence $\{u_n\}_{n\in\mathbb{Z}}$ of elements in $A_0 \cap A_1$ which satisfies (2.1) and also

$$J(\lambda^n, u_n; A_0, A_1) \le 2c(1+\lambda) K\left(\lambda^n, a; A_0, A_1\right) \ \text{ for each } n \in \mathbb{Z} \,. \tag{2.4}$$

Proof. Let us use the abbreviated notation $K(t,a)$ in place of $K(t,a;A_0,A_1)$. For each $n \in \mathbb{Z}$ there exist elements $a_n \in A_0$ and $b_n \in A_1$ such that $a = a_n + b_n$ and

$$\|a_n\|_{A_0} + \lambda^n \|b_n\|_{A_1} \le 2K(\lambda^n, a) \,. \tag{2.5}$$

We define the sequence $\{u_n\}_{n\in\mathbb{Z}}$ by setting $u_n = a_{n+1} - a_n$. Since $a_n + b_n = a_{n+1} + b_{n+1}$ we have $a_{n+1} - a_n = b_n - b_{n+1}$ and so we see that $u_n \in A_0 \cap A_1$ for each n. We also have

$$\begin{aligned} \sum_{n=-N}^{N} u_n - a &= \sum_{n=-N}^{-1} (a_{n+1} - a_n) + \sum_{n=0}^{N} (b_n - b_{n+1}) - (a_0 + b_0) \\ &= (a_0 - a_{-N}) + (b_0 - b_{N+1}) - a_0 - b_0 = -a_{-N} - b_{N+1}\,. \end{aligned} \tag{2.6}$$

So

$$\left\| \sum_{n=-N}^{N} u_n - a \right\|_{A_0+A_1} \le \|-a_{-N}\|_{A_0} + \|-b_{N+1}\|_{A_1} = \|a_{-N}\|_{A_0} + \|b_{N+1}\|_{A_1} \le 2K(\lambda^{-N}, a) + 2\lambda^{-(N+1)} \cdot K(\lambda^{N+1}, a)\,. \tag{2.7}$$

The two conditions (2.3) ensure respectively that each of the last two terms of (2.7) tends to zero as N tends to $+\infty$. Thus our sequence $\{u_n\}_{n\in\mathbb{Z}}$ satisfies (2.1).

For use in a later theorem, we introduce some additional notation, by setting

$$f_N := \sum_{n=-N}^{-1} u_n = a_0 - a_{-N} \text{ and } g_N := \sum_{n=0}^{N} u_n = b_0 - b_{N+1} \text{ for each } N \in \mathbb{N}\,. \tag{2.8}$$

It is clear from (2.6) and (2.7) that f_N and g_N are both elements of $A_0 \cap A_1$ and

$$\lim_{N\to\infty} \|f_N - a_0\|_{A_0} = \lim_{N\to\infty} \|g_N - b_0\|_{A_1} = 0\,. \tag{2.9}$$

We also have $\|f_N\|_{A_0} \le c\left(\|f_N - a_0\|_{A_0} + \|a_0\|_{A_0}\right)$ and $\|g_N\|_{A_1} \le c\left(\|g_N - b_0\|_{A_1} + \|b_0\|_{A_1}\right)$, which, together with (2.9) give

$$\sup_{N\in\mathbb{N}} \|f_N\|_{A_0} < \infty \text{ and } \sup_{N\in\mathbb{N}} \|g_N\|_{A_1} < \infty\,. \tag{2.10}$$

Finally, to obtain (2.4), we observe that, in view of (2.5) and the monotonicity of $K(t,a)$ and $\frac{1}{t}K(t,a)$ (cf. Exercise 9), we have

$$\|u_n\|_{A_0} \le c\left(\|a_{n+1}\|_{A_0} + \|a_n\|_{A_0}\right) \le 2cK(\lambda^{n+1}, a) + 2cK(\lambda^n, a) \le 2c\lambda K(\lambda^n, a) + 2cK(2^n, a) \tag{2.11}$$

and, analogously, for similar reasons,

$$\lambda^n \|u_n\|_{A_1} \le c\left(\lambda^n \|b_n\|_{A_1} + \lambda^n \|b_{n+1}\|_{A_1}\right) \le 2cK(\lambda^n, a) + 2c\frac{1}{\lambda}K(\lambda^{n+1}, a) \le 2cK(\lambda^n, a) + 2cK(\lambda^n, a)$$

which, together with (2.11) immediately gives us (2.4). □

Remark 18. (i) We can modify the preceding proof in an obvious way to get a smaller constant in (2.4) (i.e., we can replace the constant 2 in (2.5) by any other number greater than 1.) Apparently the best possible constant (or the infimum of all possible constants) is not known. However Sten Kaijser has written an ingenious paper with some much less obvious calculations for improving the constant in (2.4).

(ii) I should also mention a stronger variant of this result which I proved in [4], but only in the case of Banach couples. This result, which sometimes called the "Strong Fundamental Lemma" (or SFL for short, also maybe too pompous a name), was inspired by some remarkable results of Yuri Brudnyi and Natan Krugljak and it leads to an alternative proof of their results, and has other interesting applications (so perhaps one can after all excuse the people (not me) who chose the name SFL). In the SFL the sequence $\{u_n\}_{n\in\mathbb{Z}}$ is shown, instead of (2.4), to have the following different property

$$\sum_{n\in\mathbb{Z}} \min\left\{\|u_n\|_{A_0}, t\|u_n\|_{A_1}\right\} \le \gamma K(t, a; A_0, A_1) \text{ for all } t > 0 \tag{2.12}$$

where γ is an absolute constant. Although (2.12) is not stronger than (2.4), it is easy to use the sequence $\{u_n\}$ which satisfies (2.12) to construct another sequence $\{\widetilde{u}_n\}$ which does satisfy (2.4), but perhaps with a larger constant in place of $2c(1+\lambda)$. Several people have done further research about the SFL. My own subsequent contributions are in [5, 6, 1].

Proof of Theorem 16. Suppose that A_0 is c_0-quasinormed and A_1 is c_1-quasinormed. We define two numbers c and ρ, which depend only on c_0, c_1 and q by

$$c = \max\left\{c_0, c_1, \frac{1}{2}e^{\frac{\ln 2}{q}}\right\} \text{ and } \rho = \frac{\ln 2}{\ln 2c}$$

where we interpret $e^{\frac{\ln 2}{\infty}} = 1$. These choices of c and ρ have been made simply to ensure that:

(i) A_0 and A_1 are both c-quasinormed,

(ii) $(2c)^\rho = 2$, and

(iii) $\rho \le q$.

We will not need conditions (ii) and (iii) until the second part of the proof.

First let us suppose that a be an element of $(A_0, A_1)_{\theta,q}$. By Corollary 12 we have $a \in (A_0, A_1)_{\theta,\infty}$, from which it is clear that condition (2.3) holds. So we use the special sequence $\{u_n\}_{n\in\mathbb{Z}}$ provided by Lemma 17 to show that $a \in (A_0, A_1)_{\theta,q,J,\lambda}$ and that $\|a\|_{(A_0,A_1)_{\theta,q,J,\lambda}} \le 2c(1+\lambda)C_* \|a\|_{(A_0,A_1)_{\theta,q}}$, where C_* is a constant of equivalence between the original quasinorm of $(A_0, A_1)_{\theta,q}$ and the "discrete" quasinorm $\left(\sum_{n\in\mathbb{Z}} \left(\lambda^{-n\theta} K(\lambda^n, a; A_0, A_1)\right)^q\right)^{1/q}$ of Theorem 11.

Now we turn to proving the reverse inclusion and inequality of functionals.

But in parallel to this proof we are also going to do some other calculations which will be useful later for proving the "reiteration theorem". We will show them in red and in italics.

Suppose that $a \in (A_0, A_1)_{\theta,q,J,\lambda}$, let ϵ be an arbitrary positive number, and let $\{u_n\}_{n\in\mathbb{Z}}$ be a sequence in $A_0 \cap A_1$ which satisfies (2.1) and

$$\left(\sum_{n\in\mathbb{Z}} \left(\lambda^{-n\theta} J(\lambda^n, u_n; A_0, A_1)\right)^q\right)^{1/q} \le \|a\|_{(A_0,A_1)_{\theta,q,J,\lambda}} + \epsilon\,. \tag{2.13}$$

Since $\|a\|_{A_0}$ and $\|a\|_{A_1}$ are both c-quasinorms, it follows easily that the map $a \mapsto K(t, a; A_0, A_1)$ is a c-quasinorm on $A_0 + A_1$ for each fixed positive t. Since ρ satisfies $(2c)^\rho = 2$ we have, by Theorem 2, that the functional $a \mapsto K(t, a; A_0, A_1)^\rho$ is equivalent to a norm $K^*(t, a, A_0, A_1)$ on $A_0 + A_1$, and in fact (cf. (1.1)) that

$$K^*(t, a; A_0, A_1) \le K(t, a; A_0, A_1)^\rho \le 2K^*(t, a; A_0, A_1)\,.$$

Analogously to before, from here onwards we will usually omit "$; A_0, A_1$" in the preceding notation, thus writing $K(t, a)$ and $K^*(t, a)$ and $J(t, a)$ for the various functionals that we are using.

At the same time as we deal with the element a we also want to do some calculations for another element a_F which is defined by

$$a_F = \sum_{n\in F} u_n \tag{2.14}$$

where F is an arbitrary finite subset of $\mathbb{Z}$. It will sometimes be convenient to rewrite (2.14) as $a_F = \sum_{n=-N}^{N} \widetilde{u}_n$ where the sequence $\{\widetilde{u}_n\}_{n\in\mathbb{Z}}$ is defined by $\widetilde{u}_n = u_n$ for all $n \in F$ and $\widetilde{u}_n = 0$ for all $n \in \mathbb{Z}\backslash F$.

For each $N \in \mathbb{N}$ and each $t > 0$, we have (cf. (1.1)) that

$$\begin{aligned} K(t,a)^\rho &\le 2K^*(t,a) \le 2K^*\left(t, a - \sum_{n=-N}^{N} u_n\right) + 2K^*\left(t, \sum_{n=-N}^{N} u_n\right) \\ &\le 2K^*\left(t, a - \sum_{n=-N}^{N} u_n\right) + 2\sum_{n=-N}^{N} K^*(t, u_n) \\ &\le 2K\left(t, a - \sum_{n=-N}^{N} u_n\right)^\rho + 2\sum_{n=-N}^{N} K(t, u_n)^\rho\,. \end{aligned} \tag{2.15}$$

(

Exactly analogously to this calculation, if we choose N sufficiently large to ensure that the set F is contained in the interval $[-N, N]$, then we obtain that

$$K(t, a_F)^\rho \le 2 \sum_{n \in F} K(t, u_n)^\rho .$$

By Exercise 9 we have $K(t, b) \le \max\{t, 1\} K(t, b)$ for every $b \in A_0 + A_1$. In view of this and (2.1), if we pass to the limit as $N \to \infty$ in (2.15), the first term tends to 0 and we deduce that

$$K(t, a)^\rho \le 2 \sum_{n=-\infty}^{\infty} K(t, u_n)^\rho . \tag{2.16}$$

Next we observe, using the "trivial" decompositions $u_n = u_n + 0$ and $u_n = 0 + u_n$, that $K(t, u_n) \le \|u_n\|_{A_0} \le J(\lambda^n, u_n)$ and also $K(t, u_n) \le t \|u_n\|_{A_1} \le t\lambda^{-n} J(\lambda^n, u_n)$. So $K(t, u_n) \le \min\{1, t\lambda^{-n}\} J(\lambda^n, u_n)$. This, together with (2.16), gives us that

$$\begin{aligned} \left(\lambda^{-\theta m} K(\lambda^m, a)\right)^\rho &\le 2 \sum_{n=-\infty}^{\infty} \left(\lambda^{-\theta m} \min\{1, \lambda^{m-n}\} J(\lambda^n, u_n)\right)^\rho \\ &= 2 \sum_{n=-\infty}^{\infty} \left(\lambda^{-\theta(m-n)} \min\{1, \lambda^{m-n}\} \lambda^{-\theta n} J(\lambda^n, u_n)\right)^\rho \\ &= 2 \sum_{n=-\infty}^{\infty} \left(\min\{\lambda^{-\theta(m-n)}, \lambda^{(1-\theta)(m-n)}\} \lambda^{-\theta n} J(\lambda^n, u_n)\right)^\rho . \end{aligned}$$

Let us rewrite this inequality, setting $K_m = \left(\lambda^{-\theta m} K(\lambda^m, a)\right)^\rho$, $J_m = \left(\lambda^{-\theta m} J(\lambda^m, u_m)\right)^\rho$ and $w_m = \left(\min\{\lambda^{-\theta m}, \lambda^{(1-\theta)m}\}\right)^\rho$. We simply obtain that

$$K_m \le 2 \sum_{n=-\infty}^{\infty} w_{m-n} J_n = 2 \sum_{n=-\infty}^{\infty} w_n J_{m-n} =: 2S_m . \tag{2.17}$$

I.e., here we have defined a new numerical sequence $\{S_m\}_{m \in \mathbb{Z}}$ by setting $S_m = \sum_{n=-\infty}^{\infty} w_n J_{m-n}$ for each $m \in \mathbb{Z}$.

If we do all the calculations after (2.16) for the case where we replace $\{u_n\}_{n \in \mathbb{Z}}$ by the sequence $\{\widetilde{u}_n\}_{n \in \mathbb{Z}}$, and if we define $\widetilde{K}_m = \left(\lambda^{-\theta m} K(\lambda^m, a_F)\right)^\rho$ and $\widetilde{J}_m = \left(\lambda^{-\theta m} J(\lambda^m, \widetilde{u}_m)\right)^\rho$, then $0 \le \widetilde{J}_m \le J_m$. So, exactly analogously to (2.17), we obtain that

$$\widetilde{K}_m \le 2 \sum_{n=-\infty}^{\infty} w_n \widetilde{J}_{m-n} \le 2S_m . \tag{2.18}$$

Now, for the first time in this proof, we have to use the fact that we have chosen the constants c and ρ so that $\rho \le q$. This means that $p := q/\rho$ is in the range $[1, \infty]$ and Minkowski's inequality gives us that

$$\left(\sum_{m=-\infty}^{\infty} K_m^p\right)^{1/p} \le 2\left(\sum_{m=-\infty}^{\infty} S_m^p\right)^{1/p} \le 2 \sum_{n=-\infty}^{\infty} w_n \left(\sum_{m=-\infty}^{\infty} J_{m-n}^p\right)^{1/p} = 2 \sum_{n=-\infty}^{\infty} w_n \left(\sum_{m=-\infty}^{\infty} J_m^p\right)^{1/p} \tag{2.19}$$

or, in the case where $q = p = \infty$, that

$$\sup_{m \in \infty} K_m \le 2 \sup_{m \in \infty} S_m \le 2 \sum_{n=-\infty}^{\infty} w_n \cdot \sup_{m \in \mathbb{Z}} J_m . \tag{2.20}$$

We note that the sum

$$C_{\rho,\theta,\lambda} := \sum_{m=-\infty}^{\infty} w_m$$

is finite. It equals $1 + \frac{\lambda^{-\theta\rho}}{1-\lambda^{-\theta\rho}} + \frac{\lambda^{-(1-\theta)\rho}}{1-\lambda^{-(1-\theta)\rho}}$ if $\lambda > 1$, or $1 + \frac{\lambda^{\theta\rho}}{1-\lambda^{\theta\rho}} + \frac{\lambda^{(1-\theta)\rho}}{1-\lambda^{(1-\theta)\rho}}$ if $\lambda < 1$.

Since $\left\|\{\alpha_n\}_{n\in\mathbb{N}}\right\|_{\ell^p} = \left\|\left\{\alpha_n^{1/\rho}\right\}_{n\in\mathbb{N}}\right\|_{\ell^q}^{\rho}$ for any non negative sequence $\{\alpha_n\}$, whether or not q and p are finite, we can rewrite the estimate (2.19) if $q<\infty$, or the estimate (2.20), if $q=\infty$, to obtain

$$\|\left\{\lambda^{-\theta m}K(\lambda^m,a)\right\}_{m\in\mathbb{Z}}\|_{\ell^q}^{\rho} \le 2\|\left\{S_m^{1/\rho}\right\}_{m\in\mathbb{Z}}\|_{\ell^q}^{\rho} \le 2C_{\rho,\theta,\lambda}\|\left\{\lambda^{-\theta m}J(\lambda^m,u_m)\right\}_{m\in\mathbb{Z}}\|_{\ell^q}^{\rho}. \tag{2.21}$$

In view of (2.13) and Theorem 11, this shows that $a\in(A_0,A_1)_{\theta,q}$. Furthermore, since the positive number ϵ appearing in (2.13) can be chosen as small as we please, we deduce that $\|a\|_{(A_0,A_1)_{\theta,q}} \le C\|a\|_{(A_0,A_1)_{\theta,q,J,\lambda}}$ for some constant C depending only on θ,q,λ,c_0 and c_1. This completes the proof of the theorem, and also completes the proof begun in Exercise 14, that $\|\cdot\|_{(A_0,A_1)_{\theta,q,J,\lambda}}$ is a quasinorm. □

For later purposes we still want to use the estimate (2.21) in another way. It tells us that the sequence $\left\{S_m^{1/\rho}\right\}_{m\in\mathbb{Z}}$ is a element of ℓ^q. We also have, from (2.18), that

$$\lambda^{-\theta m}K(\lambda^m,a_F;A_0,A_1) \le 2^{1/\rho}S_m^{1/\rho} \text{ for all } m\in\mathbb{Z}. \tag{2.22}$$

Note that this estimate holds for ALL choices of the finite set F and that the numbers S_m do not depend on F.

3. The reiteration theorem for quasinormed abelian groups $(A_0,A_1)_{\theta,q}$

Theorem 19. *Let (A_0,A_1) be a quasinormed abelian couple. Then, for each θ_0, θ_1 and α in $(0,1)$ with $\theta_0\neq\theta_1$ and for each q_0, q_1 and q in $(0,\infty]$ we have*

$$\left((A_0,A_1)_{\theta_0,q_0},(A_0,A_1)_{\theta_1,q_1}\right)_{\alpha,q} = (A_0,A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q}. \tag{3.1}$$

The following two "endpoint" variants of (3.1) also hold. They correspond, roughly speaking, to the cases "$\theta_1=1$" and "$\theta_0=0$" respectively:

$$\left((A_0,A_1)_{\theta_0,q_0},A_1\right)_{\alpha,q} = (A_0,A_1)_{(1-\alpha)\theta_0+\alpha,q}. \tag{3.2}$$

$$\left(A_0,(A_0,A_1)_{\theta_1,q_1}\right)_{\alpha,q} = (A_0,A_1)_{\alpha\theta_1,q}. \tag{3.3}$$

Furthermore, in each of these three formulae the quasinorms of the groups on the left and right sides are equivalent and the constants of equivalence depend only on $\theta_0,\theta_1,\alpha,q_0,q_1,q$ and the quasinorm constants of A_0 and A_1.

Remark 20. Since the identity map of $(A_0,A_1)_{\theta,q}$ into A_0+A_1 is bounded for each $\theta\in(0,1)$ and each $q\in(0,\infty]$, we can immediately see (Cf. Remark 6) that $\left((A_0,A_1)_{\theta_0,q_0},(A_0,A_1)_{\theta_1,q_1}\right)$ and $\left((A_0,A_1)_{\theta_0,q_0},A_1\right)$ and $\left(A_0,(A_0,A_1)_{\theta_1,q_1}\right)$ are all quasinormed abelian couples. This fact should perhaps be stated explicitly as part of the formulation of Theorem 19. It is already an implicit part of the statement of the theorem, since otherwise the left sides of the formulae (3.1), (3.2) and (3.3) would not be defined.

Remark 21. One remarkable feature of this theorem is that the numbers q_0 and/or q_1 on the left side of the equations have no influence on the spaces which appear on the right side, except that they may perhaps influence the constants of equivalence between the quasinorms of the two groups. This phenomenon can be considered as a generalization of the remarkable phenomenon which occurs in the Marcinkiewicz interpolation theorem, and indeed it can be used to give us a proof of the Marcinkiewicz theorem. We will see that not only does Theorem 19 imply the formula $(L^{p_0},L^{p_1})_{\theta,q}=L^{p_\theta,q}$, but it also implies the more general formula

$$(L^{p_0,q_0},L^{p_1,q_1})_{\theta,q} = L^{p_\theta,q} \tag{3.4}$$

for all q_0,q_1,q in $(0,\infty]$, all $\theta\in(0,1)$ and all p_0,p_1 in $(0,\infty)$ with $p_0\neq p_1$. It also holds when $p_j=\infty$ for $j=0$ or $j=1$ provided that $q_j=\infty$ for that same value of j.

Remark 22. Theorem 19 was proved in a different way by Tord Holmstedt in [7]. Although he states his theorem for quasinormed spaces, the same proof probably works for quasinormed abelian groups. Theorem 19 has also been very much generalized by Yuri Brudnyi and Natan Krugljak in [3]. Per Nilsson [9] further discussed and extended some the results of [3]. Many of the results in [3] and [9] are formulated for couples of quasi–Banach or quasi normed spaces. I have not systematically checked, but I suspect that at least some of those results can be extended to the case of couples of quasinormed abelian groups.

We have already prepared most of the tools needed to prove Theorem 19. But we still need to introduce some new notions.

Definition 23. Let (A_0, A_1) be a quasinormed abelian couple and let X be a quasinormed group such that $A_0 \cap A_1 \subset X \subset A_0 + A_1$. Let θ and γ be numbers satisfying $0 \le \theta \le 1$ and $\gamma > 0$.

(i) We say that X is of class $\mathcal{C}_K(\theta, \gamma, A_0, A_1)$ if

$$K(t, a; A_0, A_1) \le \gamma t^\theta \|a\|_X \text{ for all } a \in X \text{ and all } t > 0\,.$$

(ii) We say that X is of class $\mathcal{C}_J(\theta, \gamma, A_0, A_1)$ if

$$\|a\|_X \le \gamma t^{-\theta} J(t, a; A_0, A_1) \text{ for all } a \in A_0 \cap A_1 \text{ and all } t > 0\,.$$

(iii) We say that X is of class $\mathcal{C}(\theta, \gamma, A_0, A_1)$ if it is both of class $\mathcal{C}_K(\theta, \gamma, A_0, A_1)$ and class $\mathcal{C}_J(\theta, \gamma, A_0, A_1)$.

If $\theta \in (0,1)$ then it is clear that X is of class $\mathcal{C}_K(\theta, \gamma, A_0, A_1)$ if and only if $X \subset (A_0, A_1)_{\theta,\infty}$ with $\|a\|_{(A_0,A_1)_{\theta,\infty}} \le \gamma \|a\|_X$ for all $a \in X$. Thus $(A_0, A_1)_{\theta,\infty}$ is of class $\mathcal{C}_K(\theta, 1, A_0, A_1)$. Furthermore, by Corollary 12, we also have that $(A_0, A_1)_{\theta,q}$ is of class $\mathcal{C}_K(\theta, \gamma, A_0, A_1)$ for all $q \in (0, \infty]$ for some suitable constant γ, which may depend on θ and q. As we shall show in a moment, $(A_0, A_1)_{\theta,q}$ is also of class $\mathcal{C}_J(\theta, \gamma, A_0, A_1)$ for all $\theta \in (0,1)$ and all $q \in (0, \infty]$ and a suitable constant γ depending only on θ and q. It can also apparently be shown 17 (?????) that if X is a *complete* quasinormed abelian group then it is of class $\mathcal{C}_J(\theta, \gamma, A_0, A_1)$ for some $\gamma > 0$ if and only if there exists some $q \in (0, \infty]$ such that $(A_0, A_1)_{\theta,q} \subset X$ with $\|a\|_X \le C \|a\|_{(A_0,A_1)_{\theta,q}}$ for all $a \in (A_0, A_1)_{\theta,q}$ and some suitable constant C. (This fact is stated in Remark 5.7 on p. 245 of [10]. But there the authors apparently forgot to mention that X has to be complete. The example where X is $A_0 \cap A_1$ equipped, for example, with the quasinorm of $(A_0, A_1)_{\theta,1}$ shows that the requirement of completeness cannot be removed.)

Note that in Definition 23 the parameter θ is also allowed to assume the endpoint values 0 and 1. It follows immediately from the definitions that A_0 is of class $\mathcal{C}(0, 1, A_0, A_1)$ and that A_1 is of class $\mathcal{C}(1, 1, A_0, A_1)$.

Returning to the case where $0 < \theta < 1$, let us now show, as promised above, that, for each $\theta \in (0,1)$, the group $(A_0, A_1)_{\theta,q}$ is of class $\mathcal{C}_J(\theta, \gamma, A_0, A_1)$, for some suitable γ. Indeed, for each $a \in A_0 \cap A_1$ and each positive s and t, we have

$$\begin{aligned} K(s, a; A_0, A_1) &\le \min\left\{\|a\|_{A_0}, s\|a\|_{A_1}\right\} \\ &\le \min\left\{J(t, a; A_0, A_1), \frac{s}{t} J(t, a; A_0, A_1)\right\} \\ &= J(t, a; A_0, A_1) \min\left\{1, \frac{s}{t}\right\} \end{aligned} \tag{3.5}$$

and therefore

$$\begin{aligned} \|a\|^q_{(A_0,A_1)_{\theta,q}} &\le J(t, a; A_0, A_1)^q \int_0^\infty \left(s^{-\theta} \min\left\{1, \frac{s}{t}\right\}\right)^q \frac{ds}{s} \\ &= J(t, a; A_0, A_1)^q \left(t^{-q} \int_0^t s^{q(1-\theta)} \frac{ds}{s} + \int_t^\infty s^{-q\theta} \frac{ds}{s}\right) \\ &= J(t, a; A_0, A_1)^q \left(\frac{1}{q(1-\theta)} + \frac{1}{q\theta}\right) t^{-q\theta}\,. \end{aligned}$$

This gives us that $(A_0, A_1)_{\theta,q}$ is of class $\mathcal{C}_J(\theta, \gamma, A_0, A_1)$ with $\gamma = \left(\frac{1}{q(1-\theta)} + \frac{1}{q\theta}\right)^{1/q}$ for each $q \in (0, \infty)$ and each $\theta \in (0,1)$. If $q = \infty$, we can clearly use the same estimate (3.5) to obtain the same conclusion, but this time with $\gamma = t^\theta \sup_{s>0} s^{-\theta} \min\left\{1, \frac{s}{t}\right\} = 1$.

The next lemma will give the first "half" of the proof of Theorem 19. It will deal simultaneously with the three formulae (3.1), (3.2), and (3.3). It will show that the inclusion "$\subset$" and the corresponding inequality between the respective quasinorms holds for each of them.

Lemma 24. *Let (A_0, A_1) be a quasinormed abelian couple, where both A_0 and A_1 are c-normed. Suppose that the quasinormed abelian groups X_0 and X_1 are of class $\mathcal{C}_K(\theta_0, \gamma_0, A_0, A_1)$ and $\mathcal{C}_K(\theta_1, \gamma_1 A_0, A_1)$ respectively, where the numbers θ_0 and θ_1 are in $[0,1]$ and satisfy $\theta_0 \neq \theta_1$, and γ_0 and γ_1 are positive. Then (X_0, X_1) is a quasinormed abelian couple, and, for each $\alpha \in (0,1)$ and each $q \in (0,\infty]$, we have*

$$(X_0, X_1)_{\alpha,q} \subset (A_0, A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q} \tag{3.6}$$

and

$$\|a\|_{(A_0,A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q}} \leq c\max\{\gamma_0,\gamma_1\}\,|\theta_1-\theta_0|^{-1/q}\,\|a\|_{(X_0,X_1)_{\alpha,q}} \text{ for all } a \in (X_0,X_1)_{\alpha,q}\,. \tag{3.7}$$

Proof. The conditions on X_0 and X_1 suffice (cf. Remark 6) to ensure that (X_0, X_1) is a quasinormed abelian couple. Let a be an arbitrary element of $X_0 + X_1$ and suppose that $a = x_0 + x_1$ where $x_j \in X_j$ for $j = 0, 1$. Set $\gamma = \max\{\gamma_0, \gamma_1\}$. For each fixed $t > 0$, we have that $a \mapsto K(t, a; A_0, A_1)$ is a c-quasinorm. So

$$\begin{aligned} K(t,a;A_0,A_1) &\leq c\left(K(t,x_0;A_0,A_1)+cK(t,x_1;A_0,A_1)\right) \\ &\leq c\gamma\left(t^{\theta_0}\|x_0\|_{X_0}+t^{\theta_1}\|x_1\|_{X_1}\right) = c\gamma t^{\theta_0}\left(\|x_0\|_{X_0}+t^{\theta_1-\theta_0}\|x_1\|_{X_1}\right)\,. \end{aligned}$$

If we take the infimum over all decompositions of a of the form $a = x_0 + x_1$ we deduce that

$$K(t,a;A_0,A_1) \leq c\gamma t^{\theta_0} K(t^{\theta_1-\theta_0},a;X_0,X_1) \tag{3.8}$$

and therefore, in the case where $q < \infty$, we have

$$\int_0^\infty \left(t^{-(1-\alpha)\theta_0-\alpha\theta_1}K(t,a;A_0,A_1)\right)^q \frac{dt}{t} \leq (c\gamma)^q \int_0^\infty \left(t^{-\alpha(\theta_1-\theta_0)}K(t^{\theta_1-\theta_0},a;X_0,X_1)\right)^q \frac{dt}{t}\,.$$

Since $\theta_0 \neq \theta_1$ we may make the change of variables $s = t^{\theta_1-\theta_0}$, and then the preceding integral becomes $\frac{(c\gamma)^q}{|\theta_1-\theta_0|}\int_0^\infty \left(s^{-\alpha}K(s,a;X_0,X_1)\right)^q \frac{ds}{s}$. If we take the q^{th} roots of these last three integrals, we obtain (3.7) for all $a \in (X_0, X_1)_{\alpha,q}$ (and in fact even for all $a \in X_0 + X_1$). In the remaining case, where $q = \infty$, an analogous and simpler argument enables us to deduce (3.7) from (3.8). In both cases (3.7) immediately implies (3.6) and so completes the proof. □

Now we shall turn to presenting the other "half" of the proof of Theorem 19. It will be formulated as Lemma 25. This lemma will look almost "dual" to Lemma 24, except that here, please note that we need to impose an "additional condition", either (i) or (ii). I have not carefully read the analogue of this material in [2] or in [10]. But perhaps one of these conditions, or some alternative condition, needs to be stated more explicitly in those references. As far as I can see, in [10] no such condition is imposed, and in [2] the completeness condition mentioned as part of (i) is imposed, but requirement of a "continuous embedding" of X_j into $A_0 + A_1$ is apparently not imposed. Perhaps this latter "embedding" condition can somehow be deduced from the completeness condition. At this moment I cannot see how. Perhaps one simply has to interpret the notation of the inclusion $X \subset A_0+A_1$ mentioned in the definition of spaces of class $\mathcal{C}_J(\theta, A_0, A_1)$ in [10] and in [2] as also meaning that the inclusion is a continuous mapping. The fact that the conclusion of Lemma 25 holds when the condition (ii) of the lemma is satisfied can also be apparently deduced from some more elaborate results of Tord Holmstedt [7], which although stated for the case of linear spaces rather than groups, seem to only use additive group properties of these spaces. Holmstedt in fact obtains a formula, to within equivalence, for the K-functional of the couple $\left((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1}\right)$.

Lemma 25. *Let (A_0, A_1) be a quasinormed abelian couple, where both A_0 and A_1 are c-normed. Suppose that the quasinormed abelian groups X_0 and X_1 are of class $\mathcal{C}_J(\theta_0, \gamma_0, A_0, A_1)$ and $\mathcal{C}_J(\theta_1, \gamma_1 A_0, A_1)$ respectively, where the numbers θ_0 and θ_1 are in $[0,1]$ and satisfy $\theta_0 \neq \theta_1$, and γ_0 and γ_1 are positive.*

Suppose that at least one of the following two additional conditions hold.

(i) For both $j = 0$ and $j = 1$, the quasinormed group X_j is complete, and furthermore, its quasinorm $\|\cdot\|_{X_j}$ satisfies

$$\|x\|_{A_0+A_1} \leq \alpha_j \|x\|_{X_j}^{\rho_j} \tag{3.9}$$

for all $x \in X_j$ and some positive constants α_j and ρ_j.

(ii) For both $j = 0$ and $j = 1$, the quasinormed group X_j coincides, with equality of quasinorms, with the group $(A_0, A_1)_{\theta_j,q_j}$ for some $q_j \in (0,\infty]$ if $\theta_j \in (0,1)$, or with the group A_{θ_j} if θ_j equals 0 or 1.

Then (X_0, X_1) is a quasinormed abelian couple, and, for each $\alpha \in (0,1)$ and each $q \in (0,\infty]$, we have

$$(A_0, A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q} \subset (X_0, X_1)_{\alpha,q} \tag{3.10}$$

and

$$\|a\|_{(X_0,X_1)_{\alpha,q}} \le C \|a\|_{(A_0,A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q}} \text{ for all } a \in (A_0, A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q} \,. \tag{3.11}$$

The constant C depends only on α, q, θ_0, θ_1, γ_0, γ_1 and the quasinorm constants of A_0, A_1, X_0 and X_1.

Proof. If (i) holds then (3.9) corresponds to condition (**) of Remark 6. If (ii) holds then it also implies (3.9) with $\rho_0 = \rho_1 = 1$. So in both cases we have some version of condition (**) which ensures that (X_0, X_1) is a quasinormed abelian couple.

Let a be an arbitrary element of $(A_0, A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q}$. Since $a \in (A_0, A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,\infty}$ and since $0 < (1-\alpha)\theta_0 + \alpha\theta_1 < 1$, it follows that condition (2.3) holds. So we can apply Lemma 17 with $\lambda = 2$ to obtain a sequence $\{u_n\}_{n\in\mathbb{Z}}$ of elements in $A_0 \cap A_1$ satisfying the two conditions $\lim_{N\to\infty} \left\| \sum_{n-N}^N u_n - a \right\|_{A_0+A_1} = 0$. and

$$J(2^n, u_n; A_0, A_1) \le 6cK(2^n, a; A_0, A_1) \text{ for each } n \in \mathbb{Z}\,. \tag{3.12}$$

One important step of the proof will be to use the assumption (i) or the assumption (ii) to show that the first of these two conditions can be strengthened to

$$\lim_{N\to\infty} \left\| \sum_{n-N}^{N} u_n - a \right\|_{X_0+X_1} = 0 \text{ and } a \in X_0 + X_1\,. \tag{3.13}$$

This will be rather easy if we have condition (i). But if we have condition (ii) we will need a rather longer, perhaps "messier", argument using the *red italic* material that we prepared above. So we will defer this somewhat difficult step to later, and first present the remaining part of the proof, which is a simple and even elegant application of Theorem 16:

For each $\lambda > 0$, since X_j is of class of class $\mathcal{C}_J(\theta_j, \gamma_j, A_0, A_1)$ for $j = 0, 1$, we have

$$J(\lambda^n, u_n, X_0, X_1) = \max_{j=0,1} \lambda^{jn} \|u_n\|_{X_j} \le \max_{j=0,1} \gamma_j \lambda^{jn} (2^n)^{-\theta_j} J(2^n, u_n, A_0, A_1)\,. \tag{3.14}$$

Let us choose $\lambda = 2^{\theta_1-\theta_0}$. (Recall that $\theta_0 \ne \theta_1$ so $\lambda \ne 1$.) Then the factor $\lambda^{jn}(2^n)^{-\theta_j}$ in (3.14) has the same value $\left(\lambda^{\theta_0/(\theta_1-\theta_n)}\right)^{-n}$ for both values of j. So, after we multiply both sides of (3.14) by $\lambda^{-\alpha n}$, we can deduce that

$$\lambda^{-\alpha n} J(\lambda^n, u_n, X_0, X_1) \le \left(\max_{j=0,1} \gamma_j\right) \lambda^{-\delta n} J(2^n, u_n, A_0, A_1)$$

where

$$\delta = \frac{\theta_0}{\theta_1 - \theta_0} + \alpha = \frac{\theta_0 + \alpha(\theta_1 - \theta_0)}{\theta_1 - \theta_0} = \frac{(1-\alpha)\theta_0 + \alpha\theta_1}{\theta_1 - \theta_0}$$

which means that $\lambda^{-\delta n} = 2^{-((1-\alpha)\theta_0+\alpha\theta_1)n}$. This calculation of course applies for every $n \in \mathbb{Z}$. Combining it with (3.12) we obtain that

$$\left\| \left\{ \lambda^{-\alpha n} J(\lambda^n, u_n, X_0, X_1) \right\}_{n\in\mathbb{Z}} \right\|_{\ell^q} \le 6c \left(\max_{j=0,1} \gamma_j\right) \left\| \left\{ 2^{-((1-\alpha)\theta_0+\alpha\theta_1)n} K(2^n, a; A_0, A_1 \right\}_{n\in\mathbb{Z}} \right\|_{\ell^q} \,. \tag{3.15}$$

In view of Theorem 11, the right side of (3.15) is equivalent to the quasinorm $\|a\|_{(A_0,A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q}}$ and the constants of equivalence depend only on c, γ_0, γ_1, θ_0, θ_1, α and q. If we know that (3.13) holds, then the left side of (3.15) dominates the quasinorm $\|a\|_{(X_0,X_1)_{\alpha,q,J,\lambda}}$, which, by Theorem 16 and Exercise 15, is equivalent to the quasinorm $\|a\|_{(X_0,X_1)_{\alpha,q}}$, with constants of equivalence depending only on α, q, θ_0, θ_1 and

the quasinorm constants of X_0 and X_1. Thus (3.15) will imply both (3.10) and (3.11) and complete the proof of the theorem, as soon as we have established (3.13).

Thus we come back to proving (3.13). The fact that the left side of (3.15) is finite implies that

$$\lambda^{-\alpha n} J(\lambda^n, u_n, X_0, X_1) \leq C$$

for all $n \in \mathbb{Z}$ and some constant C. Thus we have

$$\|u_n\|_{X_0} \leq C\lambda^{\alpha n} \text{ and } \|u_n\|_{X_1} \leq C\lambda^{-(1-\alpha)n}. \tag{3.16}$$

In view of Theorem 2, for a suitable positive number ρ, whose value depends on the maximum of the quasinorm constants for X_0 and X_1, we have

$$\left\| \sum_{m=m_1}^{m_2} v_m \right\|_{X_j}^{\rho} \leq 2 \sum_{m=m_1}^{m_2} \|v_m\|_{X_j}^{\rho} \tag{3.17}$$

for $j = 0, 1$ and any finite collection of elements v_m in X_j. For each $N \in \mathbb{N}$ let us consider the elements $f_N = \sum_{n=-N}^{-1} u_n$ and $g_N = \sum_{n=0}^{N} u_n$ introduced in (2.8). Suppose that $\lambda > 1$. Then, using (3.16) and (3.17), we obtain that $\{f_N\}_{n\in\mathbb{N}}$ and $\{g_N\}_{n\in\mathbb{N}}$ are Cauchy sequences in X_0 or in X_1 respectively. If $\lambda < 1$ then we have the same conclusion, but with X_0 and X_1 interchanged.

If condition (i) of the lemma holds, then each of these Cauchy sequences in X_j must converge to an element of the group X_j. Thus $f_N + g_N = \sum_{n=-N}^{N} u_n$ must converge in $X_0 + X_1$ to some element x of $X_0 + X_1$. In view of (3.9) this convergence also occurs with respect to the metric or quasinorm of $A_0 + A_1$ and so x must be the same element a of $(A_0, A_1)_{(1-\alpha)\theta_0+\alpha\theta_1,q}$ that we started with.

This completes the proof of (3.13) for the case where (i) holds. So now, suppose instead that condition (ii) holds. For the rest of this proof let us set $\theta = (1-\alpha)\theta_0 + \alpha\theta_1$. Since we have supposed that $a \in (A_0, A_1)_{\theta,q}$, we have (as already remarked and used just after (3.15)) that $\left\| \left\{ 2^{-\theta n} K(2^n, a; A_0, A_1 \right\}_{n\in\mathbb{Z}} \right\|_{\ell^q}$ is finite. By Theorem 16 it is dominated by $C \|a\|_{(A_0,A_1)_{\theta,\lambda,J,2}}$ for some constant C. Thus we have, from (3.12), that

$$\left(\sum_{n\in\mathbb{Z}} \left(2^{-n\theta} J(2^n, u_n; A_0, A_1)\right)^q \right)^{1/q} \leq 6c \left(\sum_{n\in\mathbb{Z}} \left(2^{-n\theta} K\left(2^n, a; A_0, A_1\right)\right)^q \right)^{1/q} \leq 6cC \|a\|_{(A_0,A_1)_{\theta,\lambda,J,2}}.$$

This is essentially the same as the estimate (2.13), except for the extra factor of $3cC$ here and our explicit choice of $\lambda = 2$. Thus, for any choice of the finite set $F \subset \mathbb{Z}$ and the element a_F defined by (2.14), we can use the same calculations as were done above in red to obtain the analogue of (2.22), namely

$$2^{-\theta m} K(2^m, a_F; A_0, A_1) \leq U_m \text{ for all } m \in \mathbb{Z} \tag{3.18}$$

where $\{U_m\}_{m\in\mathbb{Z}}$ is some fixed numerical sequence in ℓ^q which does not depend on the choice of the finite set F.

We will suppose that $\theta_0 < \theta_1$. Since $\alpha \in (0,1)$, it follows in this case that $\theta_0 < \theta < \theta_1$. The remaining case where $\theta_0 > \theta_1$ can be handled by analogous reasoning, or by interchanging the roles of X_0 and X_1 and using the fact (cf. Exercise 9) that $(X_0, X_1)_{\alpha,q} = (X_1, X_0)_{1-\alpha,q}$ with equality of quasinorms.

The elements $a_0 \in A_0$ and $b_0 \in A_1$ are introduced, as in Lemma 17 in the course of constructing the sequence $\{u_n\}_{n\in\mathbb{Z}}$. We will show that

$$a_0 \in X_0 \text{ and } \lim_{N\to\infty} \|f_N - a_0\|_{X_0}. \tag{3.19}$$

Analogously we will show that

$$b_0 \in X_1 \text{ and } \lim_{N\to\infty} \|g_N - b_0\|_{X_0}. \tag{3.20}$$

Since $a = a_0 + b_0$, these two conditions will suffice to establish (3.13) and complete the proof of the lemma.

In the special "endpoint" case where $\theta_0 = 0$, so that $X_0 = A_0$, we have (3.19) immediately from (2.9). Similarly, in the other endpoint case where $\theta_1 = 1$, so that $X_1 = A_1$, we obtain (3.20) also from (2.9). Thus it remains to consider the cases where $\theta_0 > 0$, so that $X_0 = (A_0, A_1)_{\theta_0,q_0}$ and where $\theta_1 < 1$, so that $X_1 = (A_0, A_1)_{\theta_1,q_1}$.

Let us now prove (3.19) in the case where $\theta_0 > 0$.

(The proof of (3.20) in the case where $\theta_1 < 1$ is analogous and will be left to the reader.)

Our first and main step will be to find convenient estimates for $K(2^n, a_0, ; A_0, A_1)$ and for $\sup_{N\in\mathbb{N}} K(2^n, f_N; A_0, A_1)$.

For each $t > 0$, we have

$$K(t, a_0; A_0, A_1) \le cK(t, f_N - a_0; A_0, A_1) + cK(t, f_N; A_0, A_1)\,.$$

Since (2.9) implies that

$$\lim_{N\to\infty} K(t, f_N - a_0; A_0, A_1) = 0 \tag{3.21}$$

we obtain that

$$K(t, a_0; A_0, A_1) \le c \lim_{N\to\infty} K(t, f_N; A_0, A_1) \;\le \sup_{N\in\mathbb{N}} K(t, f_N; A_0, A_1). \tag{3.22}$$

If we let F be the set $\{-N, -N+1, ..., -1\}$ then f_N is exactly the element a_F. So, if we let $t = 2^m$ for some $m \in \mathbb{Z}$, then we see from (3.22) and (3.18), that

$$K(2^m, f_N; A_0, A_1) \le c2^{\theta m} U_m\,.$$

The sequence $\{U_m\}_{m\in\mathbb{Z}}$ is in ℓ^q and therefore in ℓ^∞. We also have $K(2^m, f_N; A_0, A_1) \le \|f_N\|_{A_0}$. We also know (see (2.10)) that $\sup_{N\in\mathbb{N}} \|f_N\|_{A_0}$ is finite. Combining all these facts, we deduce our "main estimate", namely that

$$2^{-\theta_0 m} K(2^m, a_0; A_0, A_1) \le \sup_{N\in\mathbb{N}} 2^{-\theta_0 m} K(2^m, f_N; A_0, A_1) \le C \min\left\{2^{-\theta_0 m}, 2^{(\theta-\theta_0)m}\right\} \text{ for all } m \in \mathbb{Z}, \tag{3.23}$$

where the constant C is given by $C = \max\left\{c\left\|\{U_m\}_{m\in\mathbb{Z}}\right\|_{\ell^\infty}, \sup_{N\in\mathbb{N}}\|f_N\|_{A_0}\right\}$. Since $-\theta_0 < 0$ and $\theta - \theta_0 > 0$ we see that the sequence $\left\{\min\left\{2^{-\theta_0 m}, 2^{(\theta-\theta_0)m}\right\}\right\}_{m\in\mathbb{Z}}$ is in $\ell^p(\mathbb{Z})$ for every $p \in (0,\infty]$. So our estimate (3.23) implies that $\left\|\left\{2^{-\theta_0 m} K(2^m, a_0; A_0, A_1)\right\}_{m\in\mathbb{Z}}\right\|_{\ell^{q_0}} < \infty$ and so $a_0 \in (A_0, A_1)_{\theta_0, q_0}$, as required in the first part of (3.19).

We will now prove the second part, i.e., the convergence condition in (3.19) by using a "discrete" version of the Lebesgue dominated convergence theorem, at least in the case where $q_0 < \infty$. Here we will also use our estimate (3.23). We have that

$$\|f_N - a_0\|^{q_0}_{(A_0,A_1)_{\theta_0,q_0}} \le const. \sum_{m\in\mathbb{Z}} \left(2^{-\theta_0 m} K(2^m, f_N - a_0; A_0, A_1)\right)^{q_0}\,.$$

Each term of this series tends to 0 as $N \to \infty$ in view of (3.21). So, in order to show that the whole series tends to 0, it will suffice to show that the sequence $\left\{\left(2^{-\theta_0 m} K(2^m, f_N - a_0; A_0, A_1)\right)^{q_0}\right\}_{m\in\mathbb{Z}}$ is dominated by a sequence in ℓ^1 whose elements do not depend on N. Since A_0 and A_1 are c-quasinormed, we can use (3.23) to obtain that

$$\begin{aligned} 2^{-\theta_0 m} K(2^m, f_N - a_0; A_0, A_1) &\le 2^{-\theta_0 m} cK(2^m, f_N; A_0, A_1) + 2^{-\theta_0 m} cK(2^m, a_0; A_0, A_1) \\ &\le 2cC \min\left\{2^{-\theta_0 m}, 2^{(\theta-\theta_0)m}\right\}\,. \end{aligned} \tag{3.24}$$

Since the sequence $\left\{\left(\min\left\{2^{-\theta_0 m}, 2^{(\theta-\theta_0)m}\right\}\right)^{q_0}\right\}_{m\in\mathbb{Z}}$ is in ℓ^1 and independent of N, we are done. When $q_0 = \infty$ we can obtain the desired result from (3.21) combined with the fact that the dominating sequence in (3.24) tends to 0 as m tends to $+\infty$ and also as m tends to $-\infty$.

We have thus completed the proof of (3.19).

As already remarked, the proof of (3.20) when $\theta_1 < 1$ is analogous and will be left as an exercise.

This completes our proof of Lemma 25. □

Although we already indicated this earlier, we should perhaps state one more time that Lemma 25, when combined with Lemma 24, suffices to complete our proof of Theorem 19, because of the following facts that we showed earlier, namely that that $(A_0, A_1)_{\theta,q}$ is of class $\mathcal{C}(\theta, \gamma, A_0, A_1)$ for each $\theta \in (0,1)$ and some constant $\gamma > 0$ and that A_j is of class $\mathcal{C}(j, 1, A_0, A_1)$ for $j = 0, 1$. More precisely, since we need to invoke case (ii) in Lemma 25, the other explicit ingredients of our proof of Theorem 19 are Lemma 24, and the facts that

$(A_0, A_1)_{\theta,q}$ and A_j are, respectively, of class $\mathcal{C}_K(\theta, \gamma, A_0, A_1)$ and $\mathcal{C}_K(j, 1, A_0, A_1)$. The corresponding facts about membership of these groups in the corresponding classes $\mathcal{C}_J$ also remain relevant implicitly, because they are needed for part of the proof of Lemma 25.

4. Couples of weighted ℓ^p spaces.

In this section we will calculate the K-functional for the couple $\left(\ell^{q_0}_{\theta_0}, \ell^{q_1}_{\theta_1}\right)$ of weighted ℓ^q spaces with exponential weights. It will turn out that this calculation can be translated to also give a formula, to within equivalence, for the K-functional of the "reiterated" couple $\Big((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1}\Big)$.

For each $q \in (0, \infty]$ and each $\theta \in \mathbb{R}$ we let ℓ^q_θ denote the space of all complex valued sequences $\alpha = \{\alpha_n\}_{n\in\mathbb{Z}}$ for which the quasinorm

$$\|\{\alpha_n\}_{n\in\mathbb{Z}}\|_{\ell^q_\theta} := \left(\sum_{n\in\mathbb{Z}} \left(2^{-\theta n} |\alpha_n|\right)^q\right)^{1/q} \tag{4.1}$$

is finite. As usual, if $q = \infty$, we replace the right side of (4.1) by $\sup_{n\in\mathbb{Z}} 2^{-\theta n} |\alpha_n|$.

Theorem 26. *Suppose that $-\infty < \theta_0 < \theta_1 < \infty$ and $q_0, q_1 \in (0, \infty]$. Then the couple $\left(\ell^{q_0}_{\theta_0}, \ell^{q_1}_{\theta_1}\right)$ is a quasinormed abelian couple and, for each, $\{\alpha_n\}_{n\in\mathbb{Z}}$ in $\ell^{q_0}_{\theta_0} + \ell^{q_1}_{\theta_1}$, we have*

$$??? \le K(t, \{\alpha_n\}_{n\in\mathbb{Z}} : \ell^{q_0}_{\theta_0}, \ell^{q_1}_{\theta_1}) \le \ .$$

Proof. First we note that, for $j = 0, 1$ we obviously have $\ell^{q_j}_{\theta_j} \subset \ell^\infty_{\theta_j}$ with $\|\alpha\|_{\ell^\infty_{\theta_j}} \le \|\alpha\|_{\ell^{q_j}_{\theta_j}}$ for each $\alpha \in \ell^\infty_{\theta_j}$. Then in turn, the space $\ell^\infty_{\theta_j}$ is contained in the Banach space ℓ^∞_w of sequences $\{\alpha_n\}_{n\in\mathbb{Z}}$ for which the norm

$$\|\{\alpha_n\}_{n\in\mathbb{Z}}\|_{\ell^\infty_w} := \sup_{n\in\mathbb{Z}} \min\left\{2^{-\theta_0 n}, 2^{-\theta_1 n}\right\} |\alpha_n|$$

is finite. In fact we have $\|\alpha\|_{\ell^\infty_w} \le \|\alpha\|_{\ell^{q_j}_{\theta_j}}$ for each $\alpha \in \ell^{q_j}_{\theta_j}$ which suffices to show that $\left(\ell^{q_0}_{\theta_0}, \ell^{q_1}_{\theta_1}\right)$ is a quasinormed abelian couple.

?????$\ell^{q_0}_{\theta_0}, \ell^{q_1}_{\theta_1}$ and $\ell^{q_0}_{\theta_0}, \ell^{q_1}_{\theta_1}$ are both?????

5. Interpolation theorems for the groups $(A_0, A_1)_{\theta,q}$

The results in this section, or some slight variants of them, were already discussed in earlier lectures. It will be convenient to have the versions here on record for later use.

Let me start with a rather trivial statement.

Lemma 27. *Let (A_0, A_1) and (B_0, B_1) be two quasinormed abelian couples. Let α and β be positive constants and let T be a map from $A_0 + A_1$ into $B_0 + B_1$ which satisfies*

$$K(t, Ta; B_0, B_1) \le \alpha K(\beta t, a; A_0, A_1) \tag{5.1}$$

for all $t > 0$ and all $a \in A_0 + A_1$. Then T maps $(A_0, A_1)_{\theta,q}$ into $(B_0, B_1)_{\theta,q}$ for each $\theta \in (0, 1)$ and each $q \in (0, \infty]$, and

$$\|Ta\|_{(B_0,B_1)_{\theta,q}} \le C(\alpha, \beta, \theta, q)\|a\|_{(A_0,A_1)_{\theta,q}}$$

for each $a \in (A_0, A_1)_{\theta,q}$.

Proof. Exercise. You can also easily calculate the formula for $C(\alpha, \beta, \theta, q)$ which indeed depends only on these four parameters. $\square$

Now in the rest of this section will simply describe some other conditions on maps $T : A_0 + A_1 \to B_0 + B_1$ which are sufficient to imply (5.1). ?????

6. Some classical inequalities

You are probably quite familiar with the inequalities in the first two subsections. The inequality that we particularly need in this course is Hardy's inequality which we will meet in the third subsection.

6.1. **Hölder's inequality.**

$$\int_\Omega fg d\mu \le \left(\int_\Omega f^p d\mu\right)^{1/p} \left(\int_\Omega g^{p'} d\mu\right)^{1/p'}$$

for all measurable $f : \Omega \to [0,\infty)$ and $g : \Omega \to [0,\infty)$ and all $p \in (1,\infty)$ with $1/p + 1/p' = 1$.

Obviously equality holds if $f^p = g^{p'}$, since in that case $fg = f \cdot f^{p/p'} = f^{1+p/p'} = f^p$.

Now let f be an arbitrary function and let $\{f_n\}$ be an increasing sequence of non negative simple functions whose limit is f. Let $\{g_n\}$ be an increasing sequence of non negative simple functions defined by $g_n = (f_n)^{p/p'}$. Thus its limit is $g = f^{p/p'}$. Let $h_n = g_n / \|g_n\|_{L^{p'}}$. Of course $\|g_n\|_{L^{p'}}^{p'} = \|f_n\|_{L^p}^p$. Note that

$$\int_\Omega f_n h_n d\mu = \frac{1}{\|g_n\|_{L^{p'}}} \int_\Omega f_n g_n d\mu = \frac{1}{\|g_n\|_{L^{p'}}} \int_\Omega f_n^p d\mu = \|f_n\|_{L^p}^{p-p/p'} = \|f_n\|_{L^p}$$

so $\lim_{n\to\infty} \int_\Omega f_n h_n d\mu = \|f\|_{L^p}$ whether or not $\|f\|_{L^p}$ is finite. Since $\int_\Omega f_n h_n d\mu \le \int_\Omega f h_n d\mu \le \|f\|_{L^p}$ we see that $\lim_{n\to\infty} \int_\Omega f h_n d\mu = \|f\|_{L^p}$. So we obtain that

$$\|f\|_{L^p} = \sup_{h\in H} \int_\Omega fh d\mu \tag{6.1}$$

where H is the set of all measurable functions $h : \Omega \to [0,\infty)$ which satisfy $\|h\|_{L^p} = 1$.

6.2. **The integral form of Minkowski's inequality.**

Theorem 28. *Let (Ω, Σ, μ) and $(\Xi, \mathcal{S}, \nu)$ be two σ-finite measure spaces and let $f : \Omega \times \Xi \to [0,\infty)$ be a function which is measurable on the product measure space $(\Omega \times \Xi, \Sigma \times \mathcal{S}, \mu \times \nu)$. Then, for each $p \in [1,\infty)$,*

$$\left(\int_\Omega \left(\int_\Xi f(\omega,\xi) d\nu(\xi)\right)^p d\mu(\omega)\right)^{1/p} \le \int_\Xi \left(\int_\Omega f(\omega,\xi)^p d\mu(\omega)\right)^{1/p} d\nu(\xi)\,. \tag{6.2}$$

Proof. As Alon kindly reminded us here, we are using the Fubini or Fubini–Tonelli theorem to ensure that the integral $\int_\Xi f(\omega,\xi) d\nu(\xi)$ is well defined for μ almost every ω and defines a μ measurable function.

If $p = 1$ then (6.2) holds with equality. It is simply the Fubini or Fubini–Tonelli theorem (which may fail if the underlying measure spaces are not σ-finite).

When $p > 1$ we can calculate the left side of (6.2) by using (6.1). Let h be any function in the set H. Then, again by Fubini or Fubini–Tonelli, and then by Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega \left(\int_\Xi f(\omega,\xi) d\nu(\xi)\right) h(\omega) d\mu(\omega) &= \int_\Xi \left(\int_\Omega f(\omega,\xi) h(\omega) d\mu(\omega)\right) d\nu(\xi) \\
&\le \int_\Xi \left(\int_\Omega f(\omega,\xi)^p d\mu(\omega)\right)^{1/p} \|h\|_{L^{p'}(\mu)} d\nu(\xi) \\
&= \int_\Xi \left(\int_\Omega f(\omega,\xi)^p d\mu(\omega)\right)^{1/p} d\nu(\xi)\,.
\end{aligned}$$

Now we can obtain (6.2) simply by taking the supremum over all $h \in H$. □

Remark 29. The "usual" version of Minkowski's, namely $\|f+g\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}$, is of course a special case of the preceding theorem. It can be deduced from (6.2) by choosing Ξ to be a measure space consisting of two atoms, and can also be proved in other ways. Our assumption that the measure spaces are σ-finite was made to enable a quick proof using Fubini's theorem. In our applications here the measure spaces will be σ-finite, but you might perhaps want to wonder at some stage about whether this assumption can be removed. (If we try to do that we would of course have to add the hypothesis (or find some alternative way to ensure) that the integral $\int_\Xi f(\omega,\xi) d\nu(\xi)$ is well defined for μ almost every ω and defines a μ measurable function.)

6.3. **Hardy's inequality.**

Theorem 30. *The inequality*

$$\left(\int_0^\infty t^\alpha\left(\frac{1}{t}\int_0^t f(s)ds\right)^p dt\right)^{1/p} \le \frac{p}{p-\alpha-1}\left(\int_0^\infty t^\alpha f(t)^p dt\right)^{1/p} \tag{6.3}$$

holds for every measurable function $f:(0,\infty)\to[0,\infty)$ *and every* $p\in[1,\infty)$ *and every* $\alpha\in(-\infty,p-1)$.

Remark 31. With an appropriate interpretation this result also holds for $p=\infty$ as we shall now see. If we set $\beta=\alpha/p$ then we may rewrite (6.3) in the form

$$\left(\int_0^\infty \left(t^\beta\frac{1}{t}\int_0^t f(s)ds\right)^p dt\right)^{1/p} \le \frac{p}{p-\beta p-1}\left(\int_0^\infty \left(t^\beta f(t)\right)^p dt\right)^{1/p} \tag{6.4}$$

where now β has to satisfy $\beta\in(-\infty,1-1/p)$. If $p=\infty$ then (6.10) becomes

$$\sup_{t>0} t^{\beta-1}\int_0^t f(s)ds \le \frac{1}{1-\beta}\operatorname{ess\,sup}_{t>0} t^\beta f(t)\,. \tag{6.5}$$

This can be proved immediately by integrating the estimate $f(s)\le s^{-\beta}\operatorname{ess\,sup}_{t>0} t^\beta f(t)$.

Remark 32. In the case where f is a non increasing function an analogous inequality to (6.3) holds also when $p\in(0,1)$. But the constant $\frac{p}{p-\alpha-1}$ has to be replaced by some other constant. We shall present this result below as Theorem 33.

Proof of Theorem 30. (Cf. [8] p. 24.) For each fixed t we use the change of variable $x=s/t$ to obtain that

$$t^\alpha\left(\frac{1}{t}\int_0^t f(s)ds\right)^p = \left(\int_0^t f(s)t^{\alpha/p-1}ds\right)^p = \left(\int_0^1 f(xt)t^{\alpha/p}dx\right)^p .$$

Now we substitute this expression in the left side of (6.3) and estimate this rewritten form of the left side of (6.3) by applying (6.2) with $\Omega=(0,\infty)$ and $\Xi=(0,1)$, and with μ and ν taken to be Lebesgue measure on Ω and on Ξ respectively. This gives us

$$\left(\int_0^\infty\left(\int_0^1 f(xt)t^{\alpha/p}dx\right)^p dt\right)^{1/p} \le \int_0^1\left(\int_0^\infty\left(f(xt)t^{\alpha/p}\right)^p dt\right)^{1/p}dx$$

and then, after making the change of variable $s=xt$, this last integral equals

$$\begin{aligned}\int_0^1\left(\int_0^\infty\left(f(s)\left(\frac{s}{x}\right)^{\alpha/p}\right)^p x^{-1}ds\right)^{1/p}dx &= \int_0^1 x^{-\alpha/p-1/p}\left(\int_0^\infty f(s)^p s^\alpha ds\right)^{1/p}dx\\ &= \frac{1}{1-\alpha/p-1/p}\left(\int_0^\infty f(s)^p s^\alpha ds\right)^{1/p} .\end{aligned}$$

This completes the proof. □

Now we turn to the variant of Hardy's inequality mentioned above, for the case where $p<1$. In fact it also holds for $p=1$.

Theorem 33. *(Cf.* [8] *Theorem 2, pp. 23–25.) Each non increasing functions* $f:(0,\infty)\to[0,\infty)$ *satisfies*

$$\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx \le C(\alpha,p)\int_0^\infty f(x)^p x^\alpha dx \tag{6.6}$$

for each $p\in(0,1]$ *and each* $\alpha\in(-\infty,p-1)$*, and a constant* $C(\alpha,p)$ *depending only on* α *and* p.

Remark 34. For later reference, it will be convenient to rewrite (6.6) equivalently as

$$\left(\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx\right)^{1/p} \le C(\alpha,p)^{1/p}\left(\int_0^\infty f(x)^p x^\alpha dx\right)^{1/p} \tag{6.7}$$

One ingredient of the proof of this theorem is the (rather standard) inequality

$$\sum_{k\in\mathbb{Z}} a_k \le \left(\sum_{k\in\mathbb{Z}} a_k^p\right)^{1/p} \tag{6.8}$$

which holds for each $p \in (0,1]$ whenever the numbers a_n all satisfy $a_n \ge 0$. Its proof is a fairly straightforward exercise.

It will be convenient to also present another ingredient of the proof of this theorem separately, as the following simple and standard lemma. In fact a slight variant of this result (for non decreasing rather than non increasing functions) has already been implicitly used in the proof of Theorem 11.

Lemma 35. *For each constant $\alpha \in \mathbb{R}$ there exist contants c_α and C_α depending only on α, such that, for every non increasing function $g : (0,\infty) \to [0,\infty)$, we have*

$$c_\alpha \sum_{n\in\mathbb{Z}} g(2^n)2^{(\alpha+1)n} \le \int_0^\infty g(x)x^\alpha dx \le C_\alpha \sum_{n\in\mathbb{Z}} g(2^n)2^{(\alpha+1)n} . \tag{6.9}$$

Proof. (Cf. the first five lines of [8] p. 25.) We have

$$\begin{aligned}
\int_0^\infty g(x)x^\alpha dx &= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} g(x)x^\alpha dx \le \sum_{n\in\mathbb{Z}} \sup_{x\in[2^n,2^{n+1}]} x^\alpha \cdot \int_{2^n}^{2^{n+1}} g(2^n)dx \\
&= \max\{2^\alpha,1\} \sum_{n\in\mathbb{Z}} g(2^n)2^{\alpha n} \cdot (2^{n+1}-2^n) \\
&= \max\{2^\alpha,1\} \sum_{n\in\mathbb{Z}} g(2^n)2^{(\alpha+1)n} .
\end{aligned}$$

Conversely,

$$\begin{aligned}
\int_0^\infty g(x)x^\alpha dx &= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} g(x)x^\alpha dx \ge \sum_{n\in\mathbb{Z}} \inf_{x\in[2^n,2^{n+1}]} x^\alpha \cdot \int_{2^n}^{2^{n+1}} g(2^{n+1})dx \\
&= \min\{2^\alpha,1\} \sum_{n\in\mathbb{Z}} g(2^{n+1})2^{\alpha n} \cdot (2^{n+1}-2^n) \\
&= \min\{2^\alpha,1\} \sum_{n\in\mathbb{Z}} g(2^{n+1})2^{(\alpha+1)n} \\
&= \min\{2^\alpha,1\} \sum_{n\in\mathbb{Z}} g(2^n)2^{(\alpha+1)(n-1)} \\
&= 2^{-(\alpha+1)} \min\{2^\alpha,1\} \sum_{n\in\mathbb{Z}} g(2^n)2^{(\alpha+1)n} .
\end{aligned}$$

These two sets of inequalities establish (6.9) with $C_\alpha = \max\{2^\alpha,1\}$ and $c_\alpha = \frac{1}{2}\min\{1,2^{-\alpha}\}$. □

Proof of Theorem 33. The function $u(x) = \frac{1}{x}\int_0^x f(t)dt$ is absolutely continuous on each compact subinterval of $(0,\infty)$ and its derivative $u'(x)$ exists a.e. and is equal a.e. to $\frac{f(x)}{x} - \frac{1}{x^2}\int_0^x f(t)dt$, which is easily seen to be non positive. It follows that the function u is non increasing. Therefore the function $g(x) = \left(\frac{1}{x}\int_0^x f(t)dt\right)^p$ is also non increasing, and so we can apply Lemma 35 to obtain that

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx &\le C_\alpha \sum_{n\in\mathbb{Z}} \left(\frac{1}{2^n}\int_0^{2^n} f(t)dt\right)^p 2^{(\alpha+1)n} \\
&= C_\alpha \sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n-1} \int_{2^k}^{2^{k+1}} f(t)dt\right)^p 2^{(\alpha+1-p)n} . \qquad (6.10)
\end{aligned}$$

In view of (6.8) we have that $\sum_{k=-\infty}^{n-1}\int_{2^k}^{2^{k+1}} f(t)dt \le \left(\sum_{k=-\infty}^{n-1}\left(\int_{2^k}^{2^{k+1}} f(t)dt\right)^p\right)^{1/p}$. Thus the expression on the last line of (6.10) is dominated by the expression on the next line, which we shall then continue to estimate:

$$\begin{aligned}
& C_\alpha \sum_{n\in\mathbb{Z}}\left(\sum_{k=-\infty}^{n-1}\left(\int_{2^k}^{2^{k+1}} f(t)dt\right)^p\right)2^{(\alpha+1-p)n} \\
\le\ & C_\alpha \sum_{n\in\mathbb{Z}}\left(\sum_{k=-\infty}^{n-1}\left(\int_{2^k}^{2^{k+1}} f(2^k)dt\right)^p\right)2^{(\alpha+1-p)n} \\
=\ & C_\alpha \sum_{n\in\mathbb{Z}}\left(\sum_{k=-\infty}^{n-1} f(2^k)^p 2^{kp}\right)2^{(\alpha+1-p)n} \\
=\ & C_\alpha \sum_{k\in\mathbb{Z}}\left(\sum_{n=k+1}^{\infty} f(2^k)^p 2^{kp+(\alpha+1-p)n}\right) \\
=\ & C_\alpha \sum_{k\in\mathbb{Z}} f(2^k)^p 2^{kp}\left(\frac{2^{(\alpha+1-p)(k+1)}}{1-2^{\alpha+1-p}}\right) \\
=\ & \frac{C_\alpha\cdot 2^{\alpha+1-p}}{1-2^{\alpha+1-p}}\sum_{k\in\mathbb{Z}} f(2^k)^p 2^{k(\alpha+1)}\,.
\end{aligned}$$

The function $f(t)^p$ is of course non increasing. So we can apply Lemma 35 again, this time to show that $\sum_{k\in\mathbb{Z}} f(2^k)^p 2^{k(\alpha+1)} \le \frac{1}{c_\alpha}\int_0^\infty f(x)^p x^\alpha dx$. Combining the estimate (6.10) with all the subsequent estimates here, we obtain (6.6), where the constant $C(\alpha,p)$ satisfies

$$C(\alpha,p)=\frac{C_\alpha\cdot 2^{\alpha+1-p}}{(1-2^{\alpha+1-p})\cdot c_\alpha}=\frac{\max\{2^\alpha,1\}\cdot 2^{\alpha+1-p}}{(1-2^{\alpha+1-p})\cdot\frac{1}{2}\min\{1,2^{-\alpha}\}}=\frac{\max\{4^\alpha,1\}\cdot 2^{\alpha+2-p}}{1-2^{\alpha+1-p}}\,. \tag{6.11}$$

□

Remark 36. In the case when $p=1$ we of course have $\alpha<0$, and we can of course also invoke Theorem 30 to obtain the same estimate, but with $C(\alpha,p)=C(\alpha,1)=\frac{1}{-\alpha}$.

7. The spaces $L^{p,q}$ and their connection with the spaces $(L^p,L^\infty)_{\theta,q}$.

Let (Ω,Σ,μ) be an arbitrary measure space. We wish to study a class of spaces of (equivalence classes of) complex valued measurable functions on Ω which generalize and include the spaces $L^p(\mu)$ and Weak $L^p(\mu)$. We will define them in terms of the non increasing rearrangements f^* of measurable functions f on Ω. Thus we need to require all functions f under consideration to have distribution functions f_* which are "ultimately finite", i.e., satisfy $f_*(\alpha)<\infty$ for some sufficiently large positive number α.

Let us start by introducing a rather "crazy" space which in some kind of way is a "limiting case" of the L^p spaces as p tends to 0. It apparently appeared for the first time on pp. 248–249 of the remarkable paper [10] of Jaak Peetre and Gunnar Sparr.

For each measurable $f:\Omega\to\mathbb{C}$ define

$$\|f\|_{L^0}=\sup_{\alpha>0} f_*(\alpha)$$

and define $L^0(\mu)$ to be the set of all measurable functions $f:\Omega\to\mathbb{C}$ for which $\|f\|_{L^0}<\infty$. Clearly $\|f\|_{L^0}=\lim_{\alpha\searrow 0} f_*(\alpha)=\mu(\{\omega\in\Omega:|f(\omega)|>0\})$. So $L^0(\mu)$ consists of all measurable functions whose supports have finite measure. We will often write L^0 instead of $L^0(\mu)$ where the intended underlying measure space is clear from the context.

Exercise 37. Show that $f_*(\alpha)<\infty$ for some sufficiently large positive α if and only if $f\in L^0+L^\infty$. Show also that $\|f\|_{L^0}=\sup\{t:f^*(t)>0\}$ for all $f\in L^0+L^\infty$.

For each $p \in (0,\infty]$ and $q \in (0,\infty]$, given some underlying measure space (Ω,Σ,μ), we define $L^{p,q}$ or $L^{p,q}(\mu)$ to be the set of all (equivalence classes of) functions $f \in L^0(\mu) + L^\infty(\mu)$ for which the functional

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} \tag{7.1}$$

is finite. If $q = \infty$ then, as usual, we replace the $L^q((0,\infty), \frac{dt}{t})$ (quasi) norm in (7.1) by an essential supremum, which is in fact also simply a supremum in view of the right continuity of f^*. I.e., we have the functional

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t)\,. \tag{7.2}$$

The functionals defined in (7.1) and (7.2) are quasinorms. This is an easy consequence of the inequality

$$(f+g)^*(t) \le f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right) \text{ for all } t > 0 \tag{7.3}$$

which was discussed in an earlier document and the fact that $\|\cdot\|_{L^q}$ is a quasinorm for each $q \in (0,\infty]$. So $L^{p,q}$ is a quasinormed space and therefore also a linear space. Therefore, from now on, we can refer to $L^{p,q}$ as a space rather than just a set.

We have already met at least some of the spaces $L^{p,q}$, as we shall now clarify:

Let us first observe that, by standard properties of the non increasing rearrangement, when $p = q$ we have

$$\|f\|_{L^{p,p}} = \left(\int_0^\infty (f^*(t))^p\,dt\right)^{1/p} = \left(\int_\Omega |f|^p\,d\mu\right)^{1/p} = \|f\|_{L^p}$$

and so we have $L^{p,p} = L^p$ isometrically for all $p \in (0,\infty)$. If $p = \infty$ we define $t^{1/p} = 1$. Thus we also have $L^{\infty,\infty} = L^\infty$ isometrically. It is not hard to see that when $q < \infty$ the space $L^{\infty,q}$ contains only the zero element so we will not have anything more to say about $L^{q,\infty}$ for $q < \infty$. On the other hand when $p < \infty$ and $q = \infty$ the space $L^{p,\infty}$ coincides isometrically with the (rather important) space Weak L^p in view of an exercise that you were recently asked to do.

The spaces $L^{p,q}$ are often referred to as *Lorentz spaces.* There is indeed some overlap between these spaces and another family of spaces studied by G. G. Lorentz.

We have already noted that for $p = q \in [1,\infty]$ the quasinorm of $L^{p,q}$ is in fact a norm, and the space $L^{p,q}$ is even a Banach space. We shall see later that for all $p,q \in (1,\infty]$ this is also almost true, i.e., it is true to "within equivalence of norms". Our main tool for showing this and also for identifying the space $(L^p, L^\infty)_{\theta,q}$ will be (the two versions of) Hardy's inequality.

Let us recall that, for each measurable $f : \Omega \to \mathbb{C}$ whose distribution function satisfies $f_*(\alpha) < \infty$ for some $\alpha > 0$, we define $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$ and, more generally, $f^{**}(t,r) = \left(\frac{1}{t}\int_0^t (f^*(s))^r\,ds\right)^{1/r}$ for each $r > 0$. Let us also recall that the fact that f^* is non increasing immediately implies that

$$f^*(t) \le f^{**}(t,r) \text{ for all } t > 0\,. \tag{7.4}$$

Exercise 38. Use the formula for the K-functional for the couple $\left(L^1, L^\infty\right)$ and a general property of the K-functional of two NORMED spaces, to show that

$$(f+g)^{**}(t,r) \le f^{**}(t,r) + g^{**}(t,r) \text{ for all } t > 0 \tag{7.5}$$

for all $f, g \in L^1 + L^\infty$ in the case where $r = 1$. This inequality can be thought of as a rather nicer variant of (7.3). Prove that that $(|f|^r)^* = (f^*)^r$ for all $r > 0$ and for each $f \in L^0 + L^\infty$. Use this to extend your proof of (7.5) to the case where $1 < r < \infty$ for all $f, g \in L^0 + L^\infty$.

Now we shall see that we can essentially replace f by f^{**} or by $f^{**}(t,r)$ in the definition of the quasinorm of $L^{p,q}$.

Theorem 39. *For each $f \in L^0 + L^\infty$, the inequality*

$$\left(\int_0^\infty \left(t^{1/p} f^{**}(t,r)\right)^q \frac{dt}{t}\right)^{1/q} \le C_{p,q,r} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} \tag{7.6}$$

holds for $0 < q \le \infty$ *and* $0 < r < p$ *and some positive constant* $C_{p,q,r}$*. When* $r \le q$ *the constant* $C_{p,q,r}$ *can be taken to be* $\left(\frac{p}{p-r}\right)^{1/r}$*. When* $r \ge q$ *it can be given by* $C_{p,q,r} = (C(q/p-1, q/r))^{1/q}$ *in terms of the notation of (6.11).*

Proof. We simply have to reveal that (7.6) is a very slightly disguised version of Hardy's inequality (6.3) or of (6.5) or of the variant (6.7) of Hardy's inequality. Indeed, if we set $u(t) = (f^*(t))^r$ and raise both sides of (7.6) to the power r, then we can rewrite (7.6), in the case where q is finite, as

$$\left(\int_0^\infty t^{q/p-1}\left(\frac{1}{t}\int_0^t u(s)\right)^{q/r} dt\right)^{r/q} \le (C_{p,q,r})^r \left(\int_0^\infty t^{q/p-1}(u(t))^{q/r}\,dt\right)^{r/q} \tag{7.7}$$

So we see that here the roles of p and α in (6.3) and in (6.7) are now played in (7.7) by q/r and $q/p-1$ respectively. The condition $\alpha < p-1$, which is required for both (6.3) and (6.6), corresponds here to $q/p-1 < q/r-1$ which is equivalent to $r < p$.

When $r \le q < \infty$ then we have to invoke (6.3) and, in our context here, the constant $\frac{p}{p-\alpha-1}$ of (6.3) becomes $\frac{q/r}{q/r-q/p+1-1} = \frac{p}{p-r}$ showing that $C_{p,q,r} = \left(\frac{p}{p-r}\right)^{1/r}$.

When $q < r$, we have to invoke (6.7) and, in our context here, the constant $(C(\alpha,p))^{1/p}$ in (6.7) becomes $(C(q/p-1,q/r))^{r/q}$, which shows that $C_{p,q,r} = (C(q/p-1,q/r))^{1/q}$.

In the remaining case $q = \infty$ we have to interpret (7.6) as $\sup_{t>0} t^{1/p} f^{**}(t,r) \le \left(\frac{p}{p-r}\right)^{1/r} \sup_{t>0} t^{1/p} f^*(t)$ which, for u as above, is equivalent to

$$\sup_{t>0} t^{r/p-1}\int_0^t u(s)ds \le \frac{p}{p-r}\sup t^{r/p}u(s)$$

which is exactly (6.5) for $\beta = r/p$. $\square$

In the light of the previous theorem it makes sense to define the functional

$$\|f\|_{L^{p,q,(r)}} = \|f^{**}(\cdot,r)\|_{L^{pq}} = \left(\int_0^\infty \left(t^{1/p}f^{**}(t,r)\right)^q \frac{dt}{t}\right)^{1/q}.$$

Note that, in view of Exercise 38, this is a norm whenever $1 \le r < \infty$ and $1 \le q \le \infty$. Furthermore, in view of (7.4) and Theorem 39, we have

$$\|f\|_{L^{p,q}} \le \|f\|_{L^{p,q,(r)}} \le C_{p,r}\|f\|_{L^{p,q}} \text{ for all } f \in L^0 + L^\infty \text{ and all } r \in (0,p). \tag{7.8}$$

I.e., for all $q \in [1,\infty]$ and $p > 1$ we can make $L^{p,q}$ into a normed space by replacing its original quasinorm by an equivalent norm. For $p = 1$ we only know that $L^{1,q}$ is normable (in fact already normed) for $q = 1$.

We also mention that the space $L^{p,q}$ is complete for all values of p and q for which it is defined. But we will wait to deduce its completeness after we have identified it as an interpolation space.

We are now ready to state a general theorem about interpolation of L^p and even of $L^{p,q}$ spaces by the Lions-Peetre method.

Theorem 40. *Suppose that* p_0*,* p_1*,* q_0*,* q_1 *and* q *are in* $(0,\infty]$ *with* $p_0 \ne p_1$ *and that* θ *is in* $(0,1)$*. In the case where* $p_0 = \infty$ *or, respectively* $p_1 = \infty$*, suppose that* $q_0 = \infty$ *or, respectively, that* $q_1 = \infty$*. Then the formula*

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = L^{p_\theta,q} \tag{7.9}$$

holds to within equivalence of quasinorms, where, as usual, p_θ *is defined by the formula*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{7.10}$$

The constants of equivalence between the quasinorms depend only on $p_0, p_1, \theta, q_0, q_1$ *and* q.

Exercise 41. Does (7.9) hold when $p_0 = p_1$?

Proof of Theorem 40. Our first step will be to prove the theorem in the special case where $p_0 = q_0 < \infty$ and $p_1 = q_1 = \infty$. I.e., we have to show that

$$(L^{p_0}, L^\infty)_{\theta,q} = L^{p_\theta,q}\,. \tag{7.11}$$

We previously showed that the K-functional $K(t, f; L^p, L^\infty)$ is equivalent to $\left(\int_0^{t^p} (f^*(s))^p\, ds\right)^{1/p}$ for each $f \in L^p + L^\infty$, where the constants of equivalence depend only on p. Thus we have, for each $f \in L^r + L^\infty$, that $f^{**}(t, r)$ is equivalent to $t^{-1/r} K(t^{1/r}, f; L^r, L^\infty)$. Now, for each $p \in (r, \infty)$, we apply this equivalence together with (7.8) to obtain that $\|f\|_{L^{p,q}}$ is equivalent to

$$\left(\int_0^\infty \left(t^{1/p-1/r} K(t^{1/r}, f; L^r, L^\infty)\right)^q \frac{dt}{t}\right)^{1/q}$$

which, after the change of variables $x = t^{1/r}$ equals

$$\left(r\int_0^\infty \left(x^{r/p-1} K(x, f; L^r, L^\infty)\right)^q \frac{dx}{x}\right)^{1/q} = r^{1/p}\, \|f\|_{(L^r,L^\infty)_{1-r/p,q}}\,.$$

Thus we have shown the equality of spaces

$$L^{p,q} = (L^r, L^\infty)_{1-r/p,q} \tag{7.12}$$

to within equivalence of their quasinorms, in the case where $p \in (r, \infty)$. In fact this formula is exactly the required formula (7.11), i.e, (7.9) for $p_0 = q_0 < \infty$ and $p_1 = q_1 = \infty$. To see this, choose $p_0 = q_0 = r$ and $p_1 = q_1 = \infty$ and let θ be an arbitrary number in $(0, 1)$. Then choose $p = p_\theta$. Then it follows from (7.10) that $p_\theta = p_0/(1-\theta)$, or, equivalently, that $1 - p_0/p_\theta = \theta$. I.e., $\theta = 1 - r/p$. We also see that $r = p_0 < p_\theta = p$. So we can indeed obtain that (7.12) holds and is the same as (7.11). This completes the proof of our first step.

Our second step is to consider a slightly more general case. We still take $p_1 = q_1 = \infty$ which necessarily means that we must take $p_0 < \infty$. But now q_0 can be any element of the interval $(0, \infty]$. Thus we now have to show that

$$(L^{p_0,q_0}, L^\infty)_{\theta,q} = L^{p_\theta,q} \tag{7.13}$$

for each $\theta \in (0, 1)$ and $q \in (0, \infty]$. To do this, we choose some $p \in (0, p_0)$ and choose $\theta_0 = 1 - \frac{p}{p_0}$. Then $\theta_0 \in (0, 1)$ and $\frac{1}{p_0} = \frac{1-\theta_0}{p}$. So, by Step 1, we have that

$$L^{p_0,q_0} = (L^p, L^\infty)_{\theta_0,q_0}\,. \tag{7.14}$$

Now we choose $\theta_1 = 1$ and apply the "endpoint" formula (3.2) when (A_0, A_1) is the couple (L^p, L^∞) and $\alpha = \theta$. In this context (3.2) becomes

$$\left((L^p, L^\infty)_{\theta_0,q_0}, L^\infty\right)_{\theta,q} = (L^p, L^\infty)_{(1-\theta)\theta_0+\theta,q}\,. \tag{7.15}$$

In view of (7.14), the left sides of (7.13) and (7.15) are the same space. In view of Step 1, the right side of (7.15) is the space $L^{r,q}$, where

$$\frac{1}{r} = \frac{1-(1-\theta)\theta_0+\theta}{p} = \frac{(1-\theta)(1-\theta_0)}{p} = \frac{(1-\theta)(1-\theta_0)}{p_0(1-\theta_0)} = \frac{1-\theta}{p_0} = \frac{1}{p_\theta}\,.$$

So the right side of (7.15) equals the right side of (7.13). This establishes (7.13) and completes the proof of Step 2.

The general formula $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$ (discussed in Exercise 9) combined with the result of Step 2 enables us to immediately show that the space $(L^\infty, L^{p_1,q_1})_{\theta,q}$ equals $L^{p_\theta,q}$ where $\frac{1}{p_\theta} = 0 + \frac{\theta}{p_1}$, i.e., to prove Theorem 40 in the case where $p_0 = q_0 = \infty$.

The only remaining case is where both p_0 and p_1 are finite. Here again we will use the reiteration theorem.

Suppose that $\theta \in (0, 1)$, and $p_j \in (0, \infty)$ and $q_j \in (0, \infty]$ for $j = 0, 1$ and $q \in (0, \infty]$. We choose a number p satisfying $0 < p < \min\{p_0, p_1\}$. Then we choose numbers θ_0 and θ_1 which satisfy $\frac{1}{p_j} = \frac{1-\theta_j}{p}$, for $j = 0, 1$, i.e.,

$\theta_j = 1 - \frac{p}{p_j} \in (0,1)$. Then Step 1, i.e., the formula (7.13), gives us that $L^{p_j,q_j} = (L^p, L^\infty)_{\theta_j,q_j}$ for $j = 0,1$. So

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = \left((L^p, L^\infty)_{\theta_0,q_0}, (L^p, L^\infty)_{\theta_1,q_1}\right)_{\theta,q} .$$

This latter space can be identified using the reiteration formula (3.1), when (A_0, A_1) is the couple (L^p, L^∞). It has to coincide with the space $(L^p, L^\infty)_{(1-\theta)\theta_0+\theta\theta_1,q}$. By Step 1, this space, in turn, coincides with $L^{r,q}$ where

$$\begin{aligned} \frac{1}{r} &= \frac{1-(1-\theta)\theta_0 - \theta\theta_1}{p} = \frac{1-(1-\theta)\left(1-\frac{p}{p_0}\right) - \theta\left(1-\frac{p}{p_1}\right)}{p} \\ &= \frac{1}{p} - (1-\theta)\left(\frac{1}{p} - \frac{1}{p_0}\right) - \theta\left(\frac{1}{p} - \frac{1}{p_1}\right) = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p_\theta} . \end{aligned}$$

Thus $r = p_\theta$ and we have thus established the formula (7.9) in the one remaining case. This completes the proof of Theorem 40. $\square$

References

[1] Y. Ameur and M. Cwikel; K-divisibility constants for some special couples. arXiv:math.FA/0512234

[2] J. Bergh and J. Löfström, *Interpolation spaces. An Introduction,* Springer, Berlin 1976

[3] Y. Brudnyi and N. Krugljak, *Interpolation functors and interpolation spaces, Volume 1,* North Holland, Amsterdam 1991.

[4] M. Cwikel; K-divisibility of the K-functional and Calderón couples, Ark. Mat. 22 (1984), 39–62.

[5] M. Cwikel, B. Jawerth and M. Milman, On the fundamental lemma of interpolation theory, J. Approx. Th. 60 (1990) 70–82.

[6] M. Cwikel, The K-divisibility constant for couples of Banach lattices; J. Approx. Theory 124 (2003), 124–136.

[7] T. Holmstedt, Interpolation of quasi-normed spaces, Math. Scand. 26 (1970), 177–199.

[8] A. Kufner, L. Maligranda and L.-E. Persson, The Hardy Inequality. About its History and Some Related Results. Pilsen 2007.

[9] P. Nilsson, Reiteration theorems for real interpolation and approximation spaces, Ann. Mat. Pura ed Appl. 132 (1982), 291–330.

[10] J. Peetre and G. Sparr, Interpolation of normed Abelian groups, Ann. Mat. Pura Appl. 92 (1972), 217–262.