This examination should be done individually. Any non-human aids are permitted, but complete solutions should be included. Write your name, section-year, id-number and email address on the first page, and write your name on each of the following pages.

Solutions should be sent to algebra11@maths.lth.se or left to one of the secretaries in studieexpeditionen on the fifth floor or in my postbox at the department 10 days after you received it and latest Monday 25 April.

Complete correct solution of a problem gives maximum 1 point, partially solved problems may give some fractional points counted in 0.1 increments, and 3 points in total are required to pass.

Lycka till!

Sergei Silvestrov

1. a) Find all irreducible polynomials in $\mathbb{Z}_2[x]$ of degree at most 3. (0.4)
   b) Show that $\mathbb{K} = \mathbb{Z}_2[x]/(x^3 + x^2 + 1)$ is a field. (0.2)
   c) Let $\alpha$ be a root of $x^3 + x^2 + 1$ in $\mathbb{K}$. Find the inverse of $\alpha^6$ in $\mathbb{K}$ (0.4)

2. a) Show that the rings $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_{16}$ are not isomorphic. (0.5)
   b) Show that if $(m, n) \neq 1$ then $\mathbb{Z}_m \times \mathbb{Z}_n$ and $\mathbb{Z}_{nm}$ are non-isomorphic. (0.5)

3. a) Prove that the alternating group $A_3$ is simple.
   b) Construct complete multiplication table of the the alternating group $A_4$ and prove that $A_4$ is not simple.
   c) Find the center of the group $A_4$.
   d) Find all the subgroups of the group $A_4$ and for each subgroup find all its cosets in $A_4$. Are there normal subgroups in $A_4$ and if yes then list all of them.
   e) What is index of $A_4$ in $S_4$.

4. Let $G$ be a group, and $a, b \in G$ be such that $ab = b^2a$. Let $|a|=5$ and $b \neq e$. Find $|b|$.

5. a) Let $a, b, c$ bed positive integers.
   Prove that if $a^2 + b^2 + c^2 \equiv 0 (mod 5)$, then
      $a \equiv 0 (mod 5)$, or $b \equiv 0 (mod 5)$, or $c \equiv 0 (mod 5)$. (0.5)
   b) Prove that $\sqrt{3}$ is irrational (0.5)
6. a) Let $\mathbb{F}[x]$ be a field. Prove that $[f(x)]$ is invertible in $\mathbb{F}[x]/(p(x))$ if $p(x)$ is non-constant polynomial in $\mathbb{F}[x]$ and $f(x) \in \mathbb{F}[x]$ is relatively prime to $p(x)$. (0.5)

b) Can you find an example of a field $\mathbb{F}$ and polynomials $f(x)$ and $p(x)$ in $\mathbb{F}[x]$ such that $[f(x)]$ is not invertible in $\mathbb{F}[x]/(p(x))$? (0.5)

LYCKA TILL!