Residual-based iterations for the generalized Lyapunov equation

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We consider the generalized Lyapunov equation, defined as

$$AX + XA^T + \sum_{i=1}^{m} N_i X N_i^T + BB^T = 0, \tag{1}$$

where $A, N_1, \dots, N_m \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are given, and $X \in \mathbb{R}^{n\times n}$ is sought. This linear matrix equation appears in many applications, such as, e.g., computing Gramians for bilinear- and stochastic control systems [1, 3], as well as for certain discretizations of partial differential equations on rectangular domains. The problem of computing the solution to (1) can be intractable for medium- and large-scale problems, and methods to compute approximative solutions have been developed, see, e.g., [2, 5, 6, 7, 8].

In this work, see [4], we consider the special case where the Lyapunov operator, $\mathcal{L}(X) := AX+XA^T$, is invertible and dominant in the sense that $\rho(\mathcal{L}^{-1}\Pi) < 1$, where ρ denotes the (operator) spectral radius and $\Pi(X) := \sum_{i=1}^{m} N_i X N_i^T$. The special case considered is common in the literature, see, e.g., [2, 5, 6, 8]. Moreover, we assume that $A = A^T$ is stable (Hurwitz) and $N_i = N_i^T$ for $i = 1, \dots, m$. Under these assumptions the operator $\mathcal{M} := -\mathcal{L} - \Pi$ induces an energy norm which can be used to extend the theoretical justification for the the alternating linear scheme (ALS) presented by Kressner and Sirković in [7]. More specifically, we show that the ALS can be seen as a type of residual-based iterated model reduction and as such it produces iterates that fulfil first order necessary conditions for being $\mathcal{H}_2$-optimal. We also show a related converse, namely, that an $\mathcal{H}_2$-optimal model reduction subspace implies having a locally $\mathcal{M}$-norm optimal projection space for (1).

In analogy with ALS we show that the fixed point iteration $X_{j+1} = \mathcal{L}^{-1}(-BB^T - \Pi(X_j))$, efficiently exploited in, e.g., [8], can be seen as a residual-based iteration minimizing an upper bound of the $\mathcal{M}$-norm. Furthermore we propose a residual-based generalization of rational Krylov subspaces for computing approximations to (1). Numerical experiments are conducted to study the convergence properties of the different methods.

References

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