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An ODE framework for approximating the Gramian

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The goal of Model Order Reduction of stable linear time-invariant systems (LTI) is to replace the large and sparse system matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ of the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

by smaller ones, thereby approximately retaining the input-output behaviour of the original system. In the method Balanced Truncation i.a. the controllability Gramian

$$\mathcal{P} = \int_0^\infty e^{At} B B^\mathsf{T} e^{A^\mathsf{T} t} \, \mathrm{d}t,$$

which is equivalently characterized as the solution of the Lyapunov equation

$$A\mathcal{P} + \mathcal{P}A^\mathsf{T} + BB^\mathsf{T} = 0,$$

is utilized for this purpose. In approximate Balanced Truncation the Gramian is approximated by a low-rank matrix.

We will present and analyze a system of ODEs, whose solution for $t \to \infty$ is the Gramian. We observe that the solution evolves on a manifold and characterize numerical methods whose approximate low-rank solution evolves on this manifold as well.

In this regard we connect the method Balanced POD—derived from the integral representation of the Gramian—and the ADI method—derived from the Lyapunov equation—showing that those integration methods whose solution evolves on the manifold are equivalent to the ADI method.