Topics for today:

- Short Algebra Review
- Sources of error:
  - Condition Number & Swamping
- Partial Pivoting

Sources of Error.

- **Ill-conditioning** = sensitivity of solution to the input data (not much we can do about it)
- **Swamping** = arithmetic errors due to large numerical discrepancy in parameters (fixable)

**Ill-conditioning & Condition Number**

First we need to define the concept of norm of a vector and norm of a matrix. As usual we wish to solve the problem \( Ax = B \). Assume that \( x^* \) is an approximate solution to that problem.

**Definition.** *Infinity vector norm* (maximum vector norm): 

\[
\| x \|_\infty = \max_i |x_i|, \quad i = 1..n
\]

**Definition.** *Infinity matrix norm* (maximum matrix norm): 

\[
\| A \|_\infty = \max \text{ absolute row sum}
\]

**Definition.** *Residual*:

\[
r = B - A x^*
\]

**Definition.** *Backward Error*:

\[
\| r \|_\infty = \| B - A x^* \|_\infty
\]

**Definition.** *Forward Error*:

\[
\| x - x^* \|_\infty
\]

**Definition.** *Condition Number*: For a square matrix \( A \), \( \text{cond}(A) \) is the maximum possible error magnification factor for solving \( Ax = B \), over all possible right hand sides \( B \) (or alternatively the maximum ratio of the relative error in \( x \) over the relative error in \( B \)).

\[
\text{cond}(A) = \| A \| \cdot \| A^{-1} \|
\]

**Interpretation of Condition Number:**

The larger the condition number is then the hardest will be to solve the system \( Ax = B \), accurately.

For example if \( \text{cond}(A) = 10^k \) then we should be prepared to lose \( k \) digits of accuracy in computing the solution \( x \) when solving \( Ax = b \).

Assuming for instance double precision in the computer we may have up to 16 digits accuracy for a given number. In the case that \( \text{cond}(A) = 10^k \) then the accuracy will be reduced to \( 16-k \) digits when solving \( Ax = b \).

Example. Calculate the condition number for the following matrix \( A \):

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{, First we calculate } A^{-1} = \frac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} =
\]

Jan. 25 2019
Alexandros Sopasakis
An example: solve $Ax = B$ where

\[ A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \quad \text{First we calculate } A^{-1} = \frac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \]

Now calculate $\|A\|_\infty = 3$

and $\|A^{-1}\|_\infty = 3$, therefore $\text{cond}(A) = \|A\|\|A^{-1}\| = 9$

\[
A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}
\]

**METHOD 1**

\[
\begin{bmatrix} 10^{-20} & 1 \\ 1 & 2 \end{bmatrix} \rightarrow [\begin{bmatrix} 10^{-20} & 1 \\ 0 & 4 \end{bmatrix}]
\]

\[
\begin{bmatrix} 10^{-20} & 1 \\ 0 & 4 \end{bmatrix} \rightarrow [\begin{bmatrix} 10^{-20} & 1 \\ 0 & 4 \end{bmatrix}]
\]

Solving the 2nd equation gives $x_2 = 1$ and then $x_1 = 0$

TRUE SOLUTION IS $\begin{bmatrix} x_1 = 2 \\ x_2 = 1 \end{bmatrix}$

**METHOD 2**

Start by changing the order

\[
\begin{bmatrix} 10^{-20} & 1 \\ 1 & 2 \end{bmatrix} \rightarrow [\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}]
\]

\[
\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \rightarrow [\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}]
\]

Solving gives $x_1 = 2$, $x_2 = 1$

Clearly Method 2 is the correct method to use. The reason it worked was that initial operation where we swap row 1 with row 2. It allowed us to avoid subtracting numbers which are essentially equal.
Thus to avoid swamping problems we use **pivoting**
(that operation in the beginning in method 2)!