Lecture 4
Linear Algebra review,
Condition number,
Swamping,
Pivoting

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Vector norms

The p norm of a vector $||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

- 1-norm: $||x||_1 = \sum_{i=1}^{n} |x_i|$
- 2-norm: $\left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$
- ∞-norm: $\max_i |x_i|$

$$||x||_\infty \leq ||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2 \leq n||x||_\infty$$

Properties: $||x|| > 0$ if $x \neq 0$ $\;||cx|| = |c| \cdot ||x||$ for any scalar c
$||x + y|| \leq ||x|| + ||y||$

Matrix Norms (operator norms)

$$||A||_1 = \max_j \sum_{i=1}^{n} |a_{ij}|, \quad ||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

Properties:

1. $||A|| > 0$ if $A \neq 0$
2. $||cA|| = |c| \cdot ||A||$ for any scalar c
3. $||A + B|| \leq ||A|| + ||B||$
4. $||AB|| \leq ||A|| \cdot ||B||$
5. $||Ax|| \leq ||A|| \cdot ||x||$ for any vector x

Possible problems with solving $Ax = b$. Condition number of a system.

$Ax = b$ is ill conditioned if small perturbations in the coefficients of A or b can result in large changes in the solution x.

$$\begin{aligned} 34x + 55y &= 21 \\ 55x + 89y &= 34 \end{aligned} \qquad \Rightarrow \qquad x = -1, \; y = 1$$

Add 1% error to entry a_{21}

$$\begin{aligned} 34x + 55y &= 21 \\ 55.55x + 89y &= 34 \end{aligned} \qquad \Rightarrow \qquad x = 0.034188, \; y = 0.36068.$$

The change in (x, y) (in max-norm) is $|-1 - 0.034188| = 103.4\%$

The system is ill conditioned (since A is ill conditioned)

$$\begin{aligned} 34 \cdot (0.034188) + 55 \cdot (0.36068) &= 20.000 \\ 55.55 \cdot (0.034188) + 89 \cdot (0.36068) &= 34.000 \end{aligned}$$

Relative error in b (called residual) is negligible!

Condition number

Condition number of a matrix A for a norm, $\|\cdot\|_p$, is defined as:

$$\kappa_p(A) = ||A||_p \cdot ||A^{-1}||_p$$

If $\kappa(A) \approx 10^k$, about k significant digits will be lost in solving $Ax = b$. In the previous example, $k = 4$, so we need to have an input with a minimum of 5 correct digits so that the solution x remains useful.

Properties:

- $\kappa(A) \geq 1$
- $\kappa(I) = 1$
- $\kappa(P) = 1$
- $\kappa(cA) = \kappa(A)$
- $\kappa(D) = \frac{\max|d_i|}{\min|d_i|}$

Another problem with solving $Ax = b$: Swamping

- Using naive Gaussian elimination and start pivoting around 10^{-20} and then multiplying with 10^{20} we get the wrong solution (check it!),

$$\begin{pmatrix} 10^{-20} & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Rightarrow y = 1, \; x = 0$$

- Correct way: first row exchange then Gaussian elimination,

$$\begin{pmatrix} 1 & 2 \\ 10^{-20} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow y = 2, \; x = 1$$

What did we learn?
Multipliers should be small, pivots should be large (see next slide)
But why did we even get in trouble in the first place?
Note the discrepancy in the values of A!

Pivoting - ideas and best practices

- A pivot cannot be 0
- A small pivot can introduce large numerical errors
- Multipliers should be less than 1 (in magnitude)

Best Strategies:

- Partial pivoting chooses largest magnitude in column as pivot.
- Scaled partial pivoting chooses largest magnitude in column (**relative to the entries in its row**) as pivot.

The above can be writen as (useful for programming),

$$s_r = \max\{|a_{rp}|, |a_{rp+1}|, \ldots, |a_{rN}|\} \quad \text{for} \quad r = p : N$$

The pivotal row k is defined by

$$\frac{|a_{kp}|}{s_k} = \max\left\{\frac{|a_{pp}|}{s_p}, \frac{|a_{p+1p}|}{s_{p+1}}, \ldots, \frac{|a_{Np}|}{s_N}\right\}$$

Permutation matrices - What are they?

A permutation matrix is a matrix whose rows are permutations of the rows of the identity matrix I.

Example:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \longrightarrow \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Note how by using the matrix P the rows of A are moved according to P:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Note also a nice property of permuation matrices.
They are easily invertable! $P^{-1} = P^T$

LU decomposition with pivoting

If A is nonsingular, there is a P such that $PA = LU$
Therefore:

$$Ax = b \quad \Rightarrow \quad LUx = Pb$$

Best strategy to solve $Ax = b$:

1. Compute L, U and P
2. Compute Pb
3. Solve $Ly = Pb$ with forward substitution
4. Solve $Ux = y$ with backward substitution