Jan. 24 2019

Alexandros Sopasakis

Topics for today:

- *Gaussian Elimination & Back Substitution*
- *LU Factorization*
- *Operation Count and Complexity for each method*

Gaussian Elimination & Back Substitution

Example. Solve using Gaussian Method

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

in tableau form:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right]$$

$r_1 \cdot (-3) + r_2 \to r_2$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -7 & -7 \end{array}\right]$ elimination completed here

Starting back substitution:

$r_2\left(-\frac{1}{7}\right) \to r_2$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right]$

$r_2(-1) + r_1 \to r_1$ $\quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

$$\boxed{\text{Solution } \; x_1 = 2, \; x_2 = 1}$$

Example. Solve the same problem using LU factorization

Start by using matrix notation instead of a tableau:

$$\underset{A}{\begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x_1 & \cdots \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 & \cdots \end{bmatrix}}$$

Start by obtaining

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Start by obtaining LU factorization of A

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}$$

$r_1 \cdot (-3) + r_2 \to r_2$ $\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Done! $U = \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Check: $A = L \cdot U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}$ ✓

FACTORIZATION COMPLETED!

Solving system $AX = B$ by using $LUx = B$:

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Let this be } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

so we define $\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Solve 1st $L\,Y = B$

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ easy: $y_1 = 3$

$y_2 = 2 - 9 = -7$

So $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

Now solve the other piece $\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

That is also easy

$x_2 = 1$ and

$x_1 = 3 - 1 = 2$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ DONE!

Operation Count and Complexity for each method

Assume that we wish to solve an nxn size system: AX=B

We can count that Gaussian Elimination will require **2*n^3/3+n^2/2-7*n/6** operations
Back Substitution (after Gaussian Elimination) will require **n^2** operations

- Overall therefore the Gaussian Method requires **2*n^3/3+3*n^2/2 -7*n/6** operations to solve a problem. In general we say that the Gaussian Method takes approximately 2*n^3/3 operations or **O(n^3)** operations since that term is the largest by a long shot in the overall count of operations.

- Overall LU factorization & Back Substitution requires approximately n^3/3 + n^2 operations. In general we say that LU takes **O(n^3)** operations.

Question that would be interesting to answer: is there a best method LU or Gaussian to use for solving Ax = b and if so under which circumstances is one method better than the other?