ORDINARY DIFFERENTIAL EQUATIONS

EULER'S METHOD IS THE MOST WIDELY USED NUMERICAL METHOD TO PRODUCE APPROXIMATIONS TO THE SOLUTION OF AN ODE.

SUPPOSE $\begin{cases} y' = f(t,y) \\ y(0) = y_0 \end{cases}$

EXPLICIT EULER'S METHOD:

$$\boxed{\begin{aligned} w_{n+1} &= w_n + h f(t_n, w_n), \quad n = 0, 1, \dots \\ w_0 &= y_0 \end{aligned}}$$

where h is the step size

EULER'S IMPLICIT METHOD

$$\begin{cases} w_{n+1} = w_n + h f(t_{n+1}, w_{n+1}) \\ w_0 = y_0 \end{cases}$$

* REDUCTION OF ORDER:

STARTING, FOR EXAMPLE WITH A 3rd ORDER ODE

$$\begin{cases} y''' - 3y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = 1 \end{cases}$$

PRODUCE A SYSTEM OF 1st ORDER ODES

EXAMPLE: $y''' + 3y' - 2y = 0$ WITH $y(0) = 1$

$y'(0) = 0$

$y''(0) = 0$

PROSEDURE: LET

$u_1 = y \Rightarrow u_1' = y' = u_2 \Rightarrow u_1' = u_2$

$u_2 = y' \Rightarrow u_2' = y'' = u_3 \Rightarrow u_2' = u_3$

$u_3 = y'' \Rightarrow u_3' = y''' = -3y' + 2y \Rightarrow u_3' = -3u_2 + 2u_1$

$u_3 = y'' \Rightarrow u_3' = y''' = -3y' + 2y \Rightarrow u_3' = -3u_2 + 2u_1$

THEREFORE $\underline{u' = Au}$ WHERE $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -3 & 0 \end{pmatrix}$

INITIAL CONDITIONS:

$$\left.\begin{aligned} y(0) = 1 &\Rightarrow u_1(0) = 1 \\ y'(0) = 0 &\Rightarrow u_2(0) = 0 \\ y''(0) = 0 &\Rightarrow u_3(0) = 0 \end{aligned}\right\} \Rightarrow u(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} y' = f(t,y) \\ y(0) = y_0 \end{cases} \quad \text{IVP}$$

USUALLY UNSTABLE

EXPLICIT EULER $w_{n+1} = w_n + h f(t_n, w_n)$

IMPLICIT EULER $w_{n+1} = w_n + h f(t_{n+1}, w_{n+1})$

USUALLY STABLE

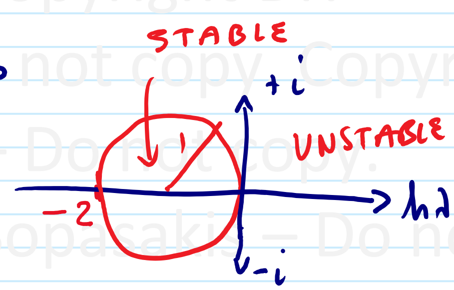
$\underline{\text{REASON}}$ if $y' = \lambda y = f(t,y)$ ← LINEAR TEST EQUATION

$\underline{\text{EXPLICIT}}$: $w_{n+1} = w_n + h\lambda w_n$

$$= (1 + h\lambda) w_n = \cdots = (1 + h\lambda)^{n+1} w_0$$

$$= (1 + h\lambda)^{n+1} y_0$$

STABLE IF (NOT TO ∞) $|1 + h\lambda| \le 1$



$\underline{\text{IMPLICIT}}$: $w_{n+1} = w_n + h\lambda w_{n+1}$

$$w_{n+1}(1 - h\lambda) = w_n \iff w_{n+1} = \frac{1}{1 - h\lambda} w_n$$

$$\ldots \; w_{n+1} = \left(\frac{1}{1 - h\lambda}\right)^{n+1} w_0 =$$

$$= \frac{1}{(1 - h\lambda)^{n+1}} y_0$$

STABLE IF (NOT TO ∞) $\left|\frac{1}{1 - h\lambda}\right| \le 1 \iff |1 - h\lambda| \ge 1$

STABLE IF
(NOT TO ∞)

$$\left|\frac{1}{1-h\lambda}\right| \leq 1 \iff |1-h\lambda| \geq 1$$