\[
\int_a^b f(x) \, dx = \sum_{k=0}^{N} \omega_k f(x_k)
\]

**IDEA: APPROXIMATE**

\[\sum_{k=0}^{N} \omega_k f(x_k)\]

- Newton-Cotes formulas
- Composite Newton-Cotes
- Romberg Integration
- Adaptive quadrature
- Gaussian quadrature

\[\int_a^b f(x) \, dx\]

**EXERCISE: APPLY TRAPEZOIDAL RULE IN ORDER TO APPROXIMATE THE AREA UNDER THE FUNCTION**

\[f(x) = \sqrt{x}\] **BETWEEN** \[1 \leq x \leq 7\]

**NOTE: EXACT AREA:**

\[
\int_1^7 \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \bigg|_1^7 = \frac{2}{3} (\sqrt{7^3} - \sqrt{1^3})
\]

\[\approx 11.68\]

**SOLUTION:**

**TRAPEZOID RULE:**

\[
\int_0^b f(x) \, dx = \frac{h}{2} \left( f(x_0) + f(x) \right) - \frac{h^3}{12} f^{(3)}(c)
\]

\[
\int_0^3 \sqrt{x} \, dx = \frac{3}{2} \left( \sqrt{3^3} + \sqrt{1} \right) - \frac{3^3}{6} \left( -\frac{1}{4} \right) 3^{3/2}
\]

\[\approx 10.937 \pm 9\] **where 9 is max error when \(3=1\).**

**EASILY THE ESTIMATE 10.9 IS WAY OFF FROM THE TRUE AREA OF 11.68**

**SIMPSON'S RULE**

\[
\int_{x_0}^{x_1} f(x) \, dx = \frac{h}{3} \left[ f(x_0) + 4f(x_1) + f(x) \right] - \frac{h^5}{90} f^{(4)}(c)
\]

\[
\approx \frac{1}{2} \left[ \frac{11}{15} \left( f(1) + 4f(3) + f(7) \right) - \frac{1}{2} \left( 15 \cdot 3^{7/2} \right) \right]
\]

**IN BLUE WE SEE THE**

**GREEN IS**

**TRAPEZOID APPROXIMATION**

**EXACT AREA**

**CLEARLY THE ESTIMATE 10.9 IS WAY OFF FROM THE TRUE AREA OF 11.68**
\[
\int_1^2 \frac{x}{x^2 + 1} \, dx = \frac{3}{3} \left[ \frac{1}{14} + \frac{9}{16} + \frac{7}{17} \right] - \frac{9}{10} \left( \frac{15.5 \text{ } 3/4}{16} \right)
\]

IN BLUE WE SEE THE SIMPSON'S APPROXIMATION (IT IS THROUGH THE PARABOLA)

\[
= \left[ \frac{148 + 17}{17} \right] - \frac{3 \cdot 15.1}{10} = 11.64 - 2.
\]

\[\text{\textbf{SOME THEORETICAL RESULTS}}\]

\textbf{FIRST THE NAME: INTEGRATION FORMULA = QUADRATURE}

\textbf{NOTE: IF THE FORMULA INCLUDES END-PTS OF INTERVAL THEN IT IS CALLED A CLOSED NEWTON-COTES QUADRATURE, OTHERWISE IT IS AN OPEN NEWTON-COTES QUADRATURE.}

\textbf{NOTE: BOTH QUADRATURES PRESENTED ABOVE ARE CLOSED NEWTON-COTES FORMULAS}

\textbf{DEF: THE ALGEBRAIC DEGREE OF ACCURACY OF A QUADRATURE FORMULA IS GIVEN BY THE POWER OF THE POLYNOMIAL P_n(x) FOR WHICH THE QUADRATURE IS EXACT (EXACT = NO ERROR AT ALL). NOA = DEGREE OF PRECISION}

\textbf{EXAMPLE (EXACT) SUPPOSE f(x) = 3 - x on [1,2] USING, FOR INSTANCE, THE TRAPEZOIDAL RULE WE GET}

\[
\int_1^2 (3-x) \, dx = \frac{1}{2} \left[ f(1) + f(2) \right] = \frac{1}{2} \left( 2 + 1 \right) = \frac{3}{2}
\]

\textbf{NOTE THAT EXACT IS:}

\[
\int_1^2 (3-x) \, dx = \left( 3x - \frac{x^2}{2} \right)_1^2 = 4 - 2.5 = 3/2
\]

\textbf{THUS THE QUADRATURE WAS EXACT FOR THIS f(x)!}

\textbf{NOTE: IF WE TRY A QUADRATIC FUNCTION THEN THE TRAPEZOIDAL RULE WILL NOT BE EXACT. SO ALGEBRAIC DEGREE OF ACCURACY = 1}

\textbf{QUESTION: WHAT IS ALGEBRAIC DEGREE OF ACCURACY FOR SIMPSON'S QUADRATURE FORMULA?}

\textbf{COMMONLY USED NEWTON-COTES QUADRATURES}

\textbf{MIDPOINT:}

\[
\int_a^b f(x) \, dx \approx \frac{b-a}{2} \left[ f(x_0) + f(x_1) \right]
\]
**Composite Quadratures**

**Composite Trapezoidal Rule:**
(Thm) Suppose \( f \in C^2[a,b] \). Then the composite rule for \( n+1 \) points, \( a = x_0, x_1, \ldots, x_n = b \) is given by:

\[
\int_a^b f(x) \, dx = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{(b-a) h^2}{12} f''(\xi)
\]

where \( h = \frac{b-a}{n} \), \( x_j = a + jh \) and \( \xi \in (a,b) \) as usual.

**Composite Midpoint Rule:**
(Thm) Suppose \( f \in C^2[a,b] \). Then the composite midpoint rule for \( n+1 \) points \( a = x_0, x_1, \ldots, x_n = b \) is given by:

\[
\int_a^b f(x) \, dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) - \frac{(b-a) h^2}{6} f''(\xi)
\]

where \( h = \frac{b-a}{n+1} \), \( x_j = a + (j+1)h \) for \( j = 0, 1, \ldots, n \) and \( \xi \in (a,b) \).

**Composite Simpson's Rule:**
(Thm) Suppose \( f \in C^4[a,b] \). Then the composite Simpson's rule for \( n+1 \) points \( a = x_0, x_1, \ldots, x_n = b \) is given by:

\[
\int_a^b f(x) \, dx = \frac{h}{3} \left[ f(a) + 4 \sum_{j=1}^{n/2} f(x_{2j}) + f(b) \right] - \frac{h^5}{90} f^{(4)}(\xi)
\]
(THM) Suppose \( f \in C^2([a,b]) \). Then the composite Simpson's rule for \( n+1 \) points \( a = x_0 < x_1 < \ldots < x_{2m} = b \) is given by

\[
\int_a^b f(x) \, dx \approx \frac{h}{3} \left[ f(a) + 2 \sum_{j=1}^{2m-1} f(x_{2j}) + 4 \sum_{j=0}^{m-1} f(x_{2j+1}) + f(b) \right] = \frac{(b-a)h^4}{180} f^{(4)}(\xi)
\]

where \( h = \frac{b-a}{2m} \), \( x_j = a + jh \) and \( \xi \in (a,b) \) as usual.

Exercise: Find the area under the curve \( f(x) = \cos(x) \) between 0 and \( \pi/2 \) using the composite midpoint rule with an error not to exceed .001.

The real question here is how many points \( n \) should we take so that the error is less than .001.

Note that the error for mid-point rule is

\[
|\text{error}| = \left| \left( \frac{x_n - x_0}{c} \right) h^2 f''(\xi) \right|
\]

where \( h = \frac{x_n - x_0}{n+2} = \frac{\pi/2 - 0}{n+2} \) and \( |f''(\xi)| = |\cos(\xi)| \leq 1 \)

Thus \( |\text{error}| \leq \frac{n^2}{6} \left( \frac{\pi/2}{n+2} \right)^2 \cdot 1 \leq .001 \)

Solving this gives \( n > \sqrt{\frac{1}{.001} \cdot \frac{\pi^3}{48}} \approx 23.41 \)

So we must choose 24 points.

Check: Using the 24 equidistant points in \([0, \pi/2]\)

we get

\[
\int_0^{\pi/2} \cos(x) \, dx \approx 2 \left( \frac{\pi/2}{24} \right) \sum_{j=0}^{11} \cos(j \pi/12) + \frac{x}{\pi} \left[ \cos(x_0) + \cos(x_2) + \ldots + \cos(x_{22}) + \cos(x_{24}) \right]
\]

\[= 1.0006 \]

Note: Exact value is \( \int_0^{\pi/2} \cos(x) \, dx = 1 \)

Actual error: \( |1 - 1.0006| = .0006 \) as we wished.

If we had chosen \( n = 23 \) then the quadrature estimate would be .99868 with an actual error of .002 (greater than what we wished).