Today.
- More examples with Deflation
  - (Review of) Complex Analysis
  - Root of unity
  - DFT (Discrete Fourier Transform)

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**Review of Deflation.**

Thin let \( A \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and associated eigenvectors \( v_1, v_2, \ldots, v_n \) and let \( x \) be any \( n \)-vector for which \( x^T v_i = 1 \). Then the matrix \( B = A - \lambda_i v_i x^T \) has eigenvalues \( 0, \lambda_2, \lambda_3, \ldots, \lambda_n \) with associated eigenvectors \( v_1, v_2, v_3, \ldots, v_n \)

\[
  x = \frac{1}{\lambda_1 v_1, k} \begin{pmatrix} a_{x1} \\ a_{x2} \\ \vdots \\ a_{xn} \end{pmatrix}
\]

A simplified version of this theorem is provided below:
Method of Deflation

1) Find \( v_1, \lambda_1 \) by Power Method

Now deflate \( A \): create new matrix \( B \) which does not have \( \lambda_1 \) as a dominant eigenvalue.

2) Deflation \[ B = A - \frac{1}{v_{1k}} v_1 A_k \]

where \( v_{1k} = k^{th} \) non-zero element of \( v_1 \) and \( A_k = k^{th} \) row of \( A \).

Note: new matrix \( B \) will have eigenvalue \( 0, \lambda_2, \lambda_3, \ldots \lambda_n \)

but different eigenvectors.

Example. \( A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \) has \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \)

\( v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

Find \( B \) and its dominant eigenvalue.

Solution: Let \( v_{1k} = v_{11} = 1 \)

So \[ B = A - \frac{1}{v_{11}} v_1 A_1 \]

\[ = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \]

\[ = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \]
Using Power Method we easily find now that B has dominant eigenvalue
\( \lambda = 2 \) which was the 2\textsuperscript{nd} largest
eigenvalue of A.

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**Fourier Transforms**

The Discrete Fourier Transform (DFT) of \( x = (x_0, \ldots, x_{n-1}) \) is

\[
-\frac{i 2\pi n}{n} x_k = \frac{1}{n} \sum_{j=0}^{n-1} x_j e^{i j \omega k}
\]

where \( w = e^{i \frac{2\pi}{n}} \).

We first need a short review of complex arithmetic

**Short Review on Complex Numbers**

- Cartesian representation
- Polar representation
- Complex conjugate
- Euler formula
- Roots of unity.

\[ \text{Def. } i = \sqrt{-1} \text{ and } z = a + ib \text{ is a complex} \]
number.

We usually visualize complex numbers as 2-dimensional vectors. We let the x-axis represent the real part of $z$ and the y-axis the imaginary part of that complex number.

**Example** Plot the following numbers:

$z_1 = 2 + 2i$

$z_2 = 3 - i$

$z_3 = -3 + 3i$

$z_4 = i$

$z = x + iy$ $i = r e^{i\theta}$

**Polar form:**

In general we have:

$$z = x + iy = r e^{i\theta}$$

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the well-known Euler formula states:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

So $z = \frac{\cos \theta + i \sin \theta}{r}$.

**Example.** Express $\sqrt{8} e^{\pi i/4}$ in the form $a + ib$.

**Solution** $\sqrt{8} e^{\pi i/4} = \sqrt{8} (\cos \pi + i \sin \pi) = 2 + 2i - z$. 
Solution  \[ \sqrt{\text{Be}^\frac{\pi}{4}} = \sqrt{\text{B}} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 2 + 2i = z_1 \]

**Def. Complex conjugate.**

The complex conjugate of \( z = a + ib \) is \( \bar{z} = a - ib \)

**Note:** \( z \cdot \bar{z} = (a + ib)(a - ib) = a^2 + iab - iab - b^2z^2 = a^2 + b^2 = |z|^2 \)

In other words \( z \cdot \bar{z} \) is always a real number.

**Example:** Complex conjugate of \( 3 - i \) is \( 3 + i \).

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**Roots of Unity:** \( z^n = 1 \)

So we need to find all \( z \) at distance 1 from origin s.t. \( z^n = 1 \).

Some examples:

\[ z^2 = 1 \]
\[ z^3 = 1 \]
\[ z^4 = 1 \]

etc...

To solve the general equation \( z^n = 1 \) we use polar
general equation $z^n = 1$ we use polar coordinates again to express both $z$ and 1.

Note: $z = e^{i\theta}$ and $1 = e^{i2\pi k}$ for $k \in \mathbb{Z}$

Thus $z^n = 1 \Rightarrow e^{in\theta} = e^{i2\pi k}$

Taking logarithms of both sides gives

$i\theta = i2\pi k$ or $\theta = \frac{2\pi k}{n}$ where $k = 0, 1, \ldots, n-1$

since if $\theta \geq n$ then we get similar angles.

So the roots of $z^n = 1$ are $z = e^{i\theta}$ with $\theta$ from above.