Lecture 1
Taylor theorem and Remainder, Errors, Bisection method

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What is numerical analysis?

"Numerical analysis is about constructing and analyzing quantitative methods for the computation of solutions to mathematical problems."

What can we solve?

A few problems can be solved *exactly:*
Linear systems

$$\begin{aligned} x + y &= 1 \\ x - y &= 0 \Rightarrow x = y = 1/2 \end{aligned}$$

Integrals of polynomials

$$I = \int_0^1 x^3 dx \Rightarrow I = 1/4$$

Most problems must be solved **approximately**

Some problems cannot be solved *exactly:*
Non-linear equations

$$x^5 + 3x^4 - 7x^3 + x^2 + 2x - 2 = 0 \qquad \Rightarrow x = ?$$

Differential equations

$$\begin{aligned} \dot{x} &= e^{t^2} \\ x(1) &= 0 \qquad \Rightarrow x(t) = ? \end{aligned}$$

Taylor's Theorem

Theorem

Let f be a $k+1$ times continuously differentiable function on the interval $[x, x_0]$ for given real numbers x and x_0. Then there exist a number ζ in the interval $[x, x_0]$ such that

$$f(x) = P_k(x) + R_k(x)$$

where the polynomial $P_k(x)$ of order k is given by,

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and the remainder (error term) is given by

$$R_k(x) = \frac{f^{(k+1)}(\zeta)}{(k+1)!}(x - x_0)^{k+1}$$

Horner's method

To evaluate $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$
at $x = c$ we use *nested multiplication:*

$$((\cdots((a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_2)x + a_1)x + a_0$$

Example: evaluate $p(t) = 1 + 4t - 5t^2 + 8t^3 - 2t^4$ at $t = 6$

$$\Rightarrow p(t) = (((-2t + 8)t - 5)t + 4)t + 1$$

1. $-2 * 6 + 8 = -4$
2. $-4 * 6 - 5 = -29$
3. $-29 * 6 + 4 = -170$
4. $-170 * 6 + 1 = -1019$

Error analysis

A "numerical solution" is different from a "mathematical solution." Numerical solutions are approximations to the exact solution.

How "good" a numerical solution is depends on how close it is to the exact solution and what is the error we are willing to tolerate. If $\hat{p}$ is an approximation to p,

- the absolute error is $E_p = |p - \hat{p}|$
- the relative error is $R_p = \frac{|p-\hat{p}|}{|p|}$. It may be expressed as a percentage.

Types of errors

- **Truncation error:** occurs when an exact formula is replaced by another to make it easier (or possible) to solve numerically.
- **Round-off error:** occurs because computers cannot represent all real numbers exactly, as computer numbers have a limited number of digits.
- **Noise:** is the error in data. The numerical result must have the same number of significant digits as the original data (same *precision*).

Notation and Accuracy

Floating point form of a number:

$$fl(p) = \pm(\frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_k}{\beta^k}) \times \beta^E$$

d_i is an integer ($0 \leq d_i \leq \beta - 1$), β is the *base*, k is the *precision*.
Normalized decimal form of a number:

$$p = \pm 0.d_1 d_2 d_3 \cdots d_k d_{k+1} \cdots \times 10^n$$

Chopping off: $fl_{chop}(p) = \pm 0.d_1 d_2 d_3 \cdots d_k \times 10^n$
Rounding off: $fl_{round}(p) = \pm 0.d_1 d_2 d_3 \cdots r_k \times 10^n$ where $d_k d_{k+1} \cdots$ is rounded to the nearest integer.

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Solving Linear and Nonlinear Equations: Bracketing methods

The annuity-due equation is $A = \frac{P}{I/12}\left((1+\frac{I}{12})^N - 1\right)$

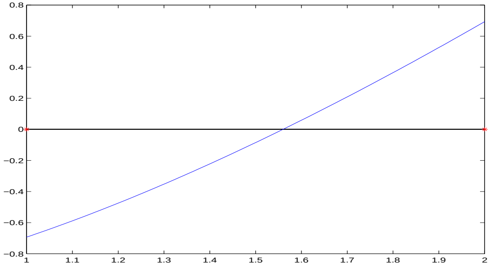
P monthly deposit, I annual interest, A amount after N deposits
You save Kr. 300 per month; what interest rate would allow you to have Kr. 50.000 after 12 years ($N = 144$)?

$$A(I) = \frac{300}{I/12}\left((1+\frac{I}{12})^{144} - 1\right) = 50.000$$

$A(0.04) = 55.331$; $A(0.03) = 51.922$; $A(0.02) = 48.779$. Answer lies in $[0.02, 0.03]$. $A(0.025) = 50.319$, answer is in $[0.020, 0.025]$. After a few more tries, $A(0.024) = 50.006$, so you must find a bank that will give you a yearly interest rate of 2.4%

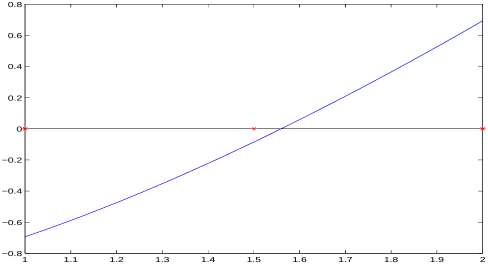
Bracketing Methods - The Bisection method

Problem: find a zero of a continuous function $f(x)$.



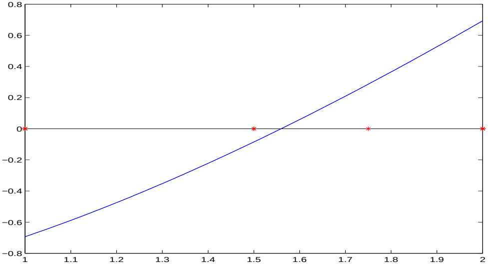
Bracketing Methods - The Bisection method

First bracket: $[1, 2]$, where $f(1) < 0$, $f(2) > 0$



First bracket: $[1, 2]$, where $f(1) < 0$, $f(2) > 0$
Midpoint: $x = 1.5$, $f(1.5) < 0 \Rightarrow [1.5, 2]$

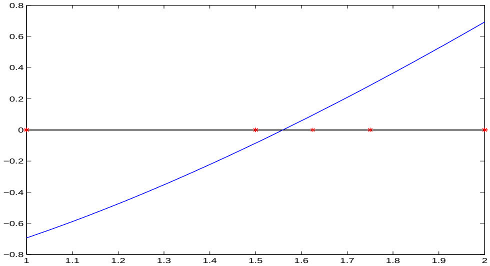
Bracketing Methods - The Bisection method



Second bracket: $[1.5, 2]$

Midpoint: $x = 1.75$, $f(1.75) > 0 \Rightarrow [1.5, 1.75]$

Bracketing Methods - The Bisection method



Third bracket: $[1.5, 1.75]$
Midpoint: $x = 1.625$, $f(1.625) > 0 \Rightarrow [1.5, 1.625]$
Approximate solution: $x = 1.5625$

Bisection theorem

Suppose

- f is continuous in $[a, b]$
- $f(r) = 0$ for some $r \in [a, b]$
- $f(a)$ and $f(b)$ have opposite signs

If $\{c_n\}$ is the sequence produced by the bisection method, then

$$|r - c_n| \leq \frac{b_n - a_n}{2} = \frac{b - a}{2^{n+1}}$$

so $\lim_{n \to \infty} c_n = r$

Example: Bisection Method to solve a nonlinear equation

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