Lecture 1
Taylor theorem and Remainder, Errors, Bisection method

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What is numerical analysis?

“Numerical analysis is about constructing and analyzing quantitative methods for the computation of solutions to mathematical problems.”
What can we solve?

A few problems can be solved exactly:

*Linear systems*

\[ \begin{align*}
  x + y &= 1 \\
  x - y &= 0 \Rightarrow x = y = 1/2
\end{align*} \]

*Integrals of polynomials*

\[ I = \int_0^1 x^3 \, dx \Rightarrow I = 1/4 \]

Most problems must be solved *approximately*.
Some problems cannot be solved exactly:

**Non-linear equations**

\[ x^5 + 3x^4 - 7x^3 + x^2 + 2x - 2 = 0 \quad \Rightarrow \quad x = ? \]

**Differential equations**

\[ \dot{x} = e^{t^2} \]

\[ x(1) = 0 \quad \Rightarrow \quad x(t) = ? \]
Taylor’s Theorem

**Theorem**

Let $f$ be a $k + 1$ times continuously differentiable function on the interval $[x, x_0]$ for given real numbers $x$ and $x_0$. Then there exist a number $\zeta$ in the interval $[x, x_0]$ such that

$$f(x) = P_k(x) + R_k(x)$$

where the polynomial $P_k(x)$ of order $k$ is given by,

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and the remainder (error term) is given by

$$R_k(x) = \frac{f^{(k+1)}(\zeta)}{(k + 1)!}(x - x_0)^{k+1}$$
Horner’s method

To evaluate $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ at $x = c$ we use nested multiplication:

$$(\cdots((a_nx + a_{n-1})x + a_{n-2})x + \cdots + a_2)x + a_1)x + a_0$$

Example: evaluate $p(t) = 1 + 4t - 5t^2 + 8t^3 - 2t^4$ at $t = 6$

$\Rightarrow p(t) = (((-2t + 8)t - 5)t + 4)t + 1$

1. $-2 \times 6 + 8 = -4$
2. $-4 \times 6 - 5 = -29$
3. $-29 \times 6 + 4 = -170$
4. $-170 \times 6 + 1 = -1019$
Error analysis

A “numerical solution” is different from a “mathematical solution.” Numerical solutions are approximations to the exact solution. How “good” a numerical solution is depends on how close it is to the exact solution and what is the error we are willing to tolerate. If \( \hat{p} \) is an approximation to \( p \),

- the absolute error is \( E_p = |p - \hat{p}| \)
- the relative error is \( R_p = \frac{|p - \hat{p}|}{|p|} \). It may be expressed as a percentage.
Types of errors

▶ **Truncation error**: occurs when an exact formula is replaced by another to make it easier (or possible) to solve numerically.

▶ **Round-off error**: occurs because computers cannot represent all real numbers exactly, as computer numbers have a limited number of digits.

▶ **Noise**: is the error in data. The numerical result must have the same number of significant digits as the original data *(same precision)*.
Notation and Accuracy

Floating point form of a number:

\[ fl(p) = \pm \left( \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_k}{\beta^k} \right) \times \beta^E \]

\( d_i \) is an integer \( (0 \leq d_i \leq \beta - 1) \), \( \beta \) is the base, \( k \) is the precision.

Normalized decimal form of a number:

\[ p = \pm 0.d_1d_2d_3\cdots d_k d_{k+1}\cdots \times 10^n \]

Chopping off: \( fl_{chop}(p) = \pm 0.d_1d_2d_3\cdots d_k \times 10^n \)

Rounding off: \( fl_{round}(p) = \pm 0.d_1d_2d_3\cdots r_k \times 10^n \) where \( d_k d_{k+1}\cdots \) is rounded to the nearest integer.
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Solving Linear and Nonlinear Equations: Bracketing methods

The annuity-due equation is

\[ A = \frac{P}{I/12} \left( (1 + \frac{I}{12})^N - 1 \right) \]

\( P \) monthly deposit, \( I \) annual interest, \( A \) amount after \( N \) deposits

You save Kr. 300 per month; what interest rate would allow you to have Kr. 50.000 after 12 years (\( N = 144 \))?

\[ A(I) = \frac{300}{I/12} \left( (1 + \frac{I}{12})^{144} - 1 \right) = 50.000 \]

\( A(0.04) = 55.331; \ A(0.03) = 51.922; \ A(0.02) = 48.779. \) Answer lies in [0.02, 0.03]. \( A(0.025) = 50.319, \) answer is in [0.020, 0.025]. After a few more tries, \( A(0.024) = 50.006, \) so you must find a bank that will give you a yearly interest rate of 2.4%
Bracketing Methods - The Bisection method

**Problem:** find a zero of a continuous function $f(x)$. 
Bracketing Methods - The Bisection method

First bracket: $[1, 2]$, where $f(1) < 0$, $f(2) > 0$

Midpoint: $x = 1.5$, $f(1.5) < 0 \Rightarrow [1.5, 2]$
Bracketing Methods - The Bisection method

Second bracket: [1.5, 2]
Midpoint: \( x = 1.75, \ f(1.75) > 0 \Rightarrow [1.5, 1.75] \)
Bracketing Methods - The Bisection method

Third bracket: [1.5, 1.75]
Midpoint: $x = 1.625$, $f(1.625) > 0 \Rightarrow [1.5, 1.625]$
Approximate solution: $x = 1.5625$
Bisection theorem

Suppose

- $f$ is continuous in $[a, b]$
- $f(r) = 0$ for some $r \in [a, b]$
- $f(a)$ and $f(b)$ have opposite signs

If \( \{c_n\} \) is the sequence produced by the bisection method, then

\[
|r - c_n| \leq \frac{b_n - a_n}{2} = \frac{b - a}{2^{n+1}}
\]

so

\[\lim_{n \to \infty} c_n = r\]
Example: Bisection Method to solve a nonlinear equation

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