Numerical Analysis — FMN011 — Omtenta 2017/08/14

The exam lasts 5 hours and has 15 questions. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those you obtained in your two home assignments, and the final grade is based on your total score.

**Justify all your answers and write down all important steps.** Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. (4p) The following procedure finds the simple root of \( f(x) = 0 \) in the interval \([a, b]\) using the bisection method so that the absolute error is less than the given positive quantity \( \delta \). Some parts are missing; fill in the blanks.

   1. Set \( a^{(1)} = a, b^{(1)} = b \), and \( i = 0 \).
   2. \( i = i + 1 \).
   3. Calculate \( x_m = \ldots \).
   4. If \( |x_m - \hat{x}| \leq \delta \), take the desired root as \( \hat{x} = \ldots \) and stop. Otherwise, continue to next step.
   5. If \( f(a^{(i)}) \cdot f(x_m) > 0 \), set \( a^{(i+1)} = \ldots \) and \( b^{(i+1)} = \ldots \), and go to step 2.
   6. If \( f(a^{(i)}) \cdot f(x_m) < 0 \), set \( a^{(i+1)} = \ldots \) and \( b^{(i+1)} = \ldots \), and go to step 2.

2. (4p) We want to find the zero of \( f(x) = x^3 + 2x - 2 \) that is close to 0.77 using a fixed point iteration.

   (a) Show that the root of \( f \) is the fixed point of \( g(x) = 2/(x^2 + 2) \).
   (b) Explain why the fixed point iteration \( x_{n+1} = g(x_n) \) will converge to the zero of \( f \) (do not carry out the iteration).
   (c) Explain why the iteration will not converge to the zero of \( f \) if we take \( g(x) = 2 - x^3 - x \).
   (d) Explain why the iteration will not converge to the zero of \( f \) if we take \( g(x) = 2 - x^3 + x \).

3. (4p) We wish to find the negative root of \( x^3 + 3x^2 - 4 = 0 \) with Newton-Raphson’s method, \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \).

   (a) Write the Newton-Raphson iteration for this problem, making sure the indices of your formula are correct.
   (b) Find the negative root of the given equation with an accuracy of 4 decimals.
(c) Calculate the rate of convergence of the iterative process.
(d) Explain if and how this rate can be accelerated.

4. (4p) Explain why we cannot define the norm of a matrix as the product of its diagonal elements.

5. (4p) Suppose the condition number of a matrix $A$ is $10^9$. Explain why we cannot “fix” the condition number by multiplying $A$ by a small number $\epsilon$.

6. (5p) Consider the system of equations
   \begin{align*}
   x_1 + x_2 &= 1 \\
   x_1 - x_2 &= 2 \\
   4x_1 + x_2 &= 3
   \end{align*}

   (a) Write the normal equations for this system.
   (b) Calculate the least squares solution to the system.
   (c) Calculate the residual vector and its 2-norm.
   (d) Is there a different solution that would give a residual with 2-norm equal to 1? Justify.
   (e) Give a basis for the space that is orthogonal to the residual vector.

7. (5p) Construct an interpolating polynomial of degree 3 for $f(x) = \sin(x/2)$ on the interval $[-\pi, \pi]$ using equally spaced data points.

8. (5p) Give an entire set of conditions that define a quadratic spline with knots $(-1,1), (0,1/2), (1/3,1), (1,2)$.

9. (5p) Draw a sketch of the Bézier curve with control points $(0,0), (0.5,-0.5), (0.5,0.5), (1,0.5)$. Mark the control points and the control polygon.

10. (5p) Which are the 8-th roots of unity? Which of them are primitive?

11. (5p) We wish to use the DFT interpolation theorem in $[0, \pi]$ to interpolate $f(t) = |t - \pi/2|$ with 4 interpolation points.
   (a) Give the set of interpolation points $(t_j, f(t_j))$.
   (b) Knowing that
       \[ F_4 = \frac{1}{2} \begin{pmatrix}
       1 & 1 & 1 & 1 \\
       1 & -i & -1 & i \\
       1 & -1 & 1 & -1 \\
       1 & i & -1 & -i
       \end{pmatrix}, \]
   calculate the Fourier transform of $x = [f(t_0), f(t_1), f(t_2), f(t_3)]^T$.
   (c) How many operations does it take to calculate the Fourier transform for a 4-dimensional real vector $x$? How many operations would it take if you used the fast Fourier transform?
   (d) Construct the interpolating polynomial given by the formula
       \[ P_n(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} \left( a_k \cos \frac{2k\pi(t-c)}{d-c} - b_k \sin \frac{2k\pi(t-c)}{d-c} \right) + \frac{a_{n/2}}{\sqrt{n}} \cos \frac{n\pi(t-c)}{d-c}. \]
(c) Evaluate $P_n(\pi)$. What is the relative error (in %) of the interpolation at this point?

12. (5p) The DFT of a real vector is

\[-0.15\]
\[-0.25 + 3.59i\]
\[-0.35 - 0.92i\]
\[-0.45 - 0.22i\]
\[-0.55\]
\[-0.45 + 0.22i\]
\[-0.35 + 0.92i\]
\[-0.25 - 3.59i\]

Given that the DFT trigonometric interpolation polynomial is

\[P_n(t) = a_0 \sqrt{n} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} (a_k \cos(2\pi kt) - b_k \sin(2\pi kt)) + a_{n/2} \sqrt{n} \cos(n\pi t),\]

describe how to construct a vector of 32 entries in order to plot the trigonometric polynomial. Describe how to use the vector in order to plot the trigonometric polynomial.

13. (5p) Suppose I want a real trigonometric polynomial that interpolates the points (0,0), (0.25,0.71),(0.5,1),(0.75,0.71). Give me instructions on how to do it.

14. (5p) Suppose you have a 64 × 64 intensity matrix corresponding to a black-and-white image. Enumerate the (five) steps you need to carry out to compress it using the JPEG standard.

15. (5p) The DC component of an 8 × 8 transformed and quantized image matrix is given in JPEG code as 100010. Decode this entry using Figure 1.

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