Repetition questions for Functional Analysis and Harmonic Analysis

The course is defined by Renardy-Rogers chapters 6-8.4. Furthermore the Weierstrass approximation theorem and the Arzela - Ascoli theorem. To receive pass on the oral exam you should do well in the basic questions below. To receive a high grade you should do well on most of the questions below.

**Basic questions**

1. Give the definition of a normed linear space, Banach and Hilbert spaces.
2. Give examples of Banach and Hilbert spaces.
3. Prove that a linear operator between two normed linear spaces is continuous if and only if it is bounded.
4. State and prove the projection theorem in Hilbert spaces.
5. Give the definition of the dual space of a Banach space.
7. Give the definition of weak and weak∗ convergence in Banach spaces.
8. Give the definition of the Sobolev spaces $W^{k,p}(\Omega)$.
9. Give the definition of the Fourier transform on $S(\mathbb{R}^n)$.
10. Give the definition in terms of the Fourier transform of the Sobolev spaces $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$ and use this to generalize the definition to non integers values.
11. Give the definition of a compact imbedding.
12. State the Poincaré inequality.
13. State the Arzela-Ascoli theorem.
14. Give the Neumann series for the inverse of $I - A$ if $\|A\| < 1$.
15. Define the spectrum of an operator in $\mathcal{L}(B, B)$ where $B$ is a Banach space.
16. Give the definition of the Banach and Hilbert space adjoints of an operator.

**Advanced questions**

1. Sketch the proof of the fact that a normed linear space always can be densely and continuously imbedded in a complete normed linear space.
2. State and prove Hölder's and Minkowski's inequalities.
3. Give example of different ON-basis for $L^2([0, 1])$. 
(4) State and prove Parseval’s formula (equality) for a given ON-basis in a Hilbert space.

(5) State and prove the closest point property in a Hilbert space.

(6) State and prove the Poincaré inequality.

(7) State and prove the Weierstrass approximation theorem.

(8) State Alaoglu’s theorem of weak compactness.

(9) State and prove the Arzela-Ascoli theorem.

(10) Prove that \( H^s(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n) \) is a continuous imbedding if \( s > n/2 \).

(11) If \( X, Y \) are Banach spaces and \( (\mathcal{D}(A), A) \) is a linear, bounded and densely defined operator from \( X \) to \( Y \), prove that \( A \) can be uniquely extended to a linear bounded operator defined on all of \( X \).

(12) State and prove the Hahn-Banach theorem on Hilbert spaces.

(13) Show that the spectrum of a bounded linear operator is a compact subset of \( \mathbb{C} \).