Fourier transform

For a function \( f \in L^1(\mathbb{R}^n) \) define the Fourier transform is defined by

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} \, dx.
\]

The Fourier transform of a function \( f \in C_0^\infty(\mathbb{R}^n) \) has many good properties. It is smooth and bounded and considered as a function of complex variables it is even an entire analytic function. However it is not evident that repeated Fourier transforms exist. It is often more convenient to use the following class of functions.

The Schwartz space \( \mathcal{S} \) is defined by

\[
\mathcal{S}(\mathbb{R}^n) = \{ f \in C_0^\infty(\mathbb{R}^n) ; |x^\alpha \partial^\beta f(x)| \text{ is bounded for every } \alpha \text{ and } \beta \}\.
\]

Then, if \( f \in \mathcal{S}(\mathbb{R}^n) \), then \( \hat{f} \in \mathcal{S}(\mathbb{R}^n) \).

**Exercise 1** For which \( f \in C_0^\infty(\mathbb{R}^n) \) is \( \hat{f} \in C_0^\infty(\mathbb{R}^n) \)?

**Exercise 2** Find the Fourier transform of the function \( e^{-x^2} \in C^\infty(\mathbb{R}) \).

**Exercise 3** Find the Fourier transform of \( e^{-|x|} \) and \((1 + x^2)^{-1} \) both in \( C^\infty(\mathbb{R}) \).

**Exercise 4** Show that

\[
\int_{\mathbb{R}^n} (1 + |x|)^{-s} \, dx
\]

is convergent if \( s > n \).

**Exercise 5** Show that \(|x|f(x)\) is bounded if \( f \in \mathcal{S} \).

**Exercise 6** Show that \((1 + |x|) \sim (1 + |x|^2)^{1/2} \) where the notation \( f \sim g \) means that there exist positive constants \( c_1, c_2 \) such that \( c_1 \leq f/g \leq c_2 \).

**Exercise 7** If \( s \geq 0 \), show that for any \( k \) with \( 0 \leq k \leq s \) there exists a positive constant \( C_k \) such that \(|x|^k \leq C_k (1 + |x|^2)^{s/2} \). Also show that if \( \beta \) is a multi-index with \( 0 \leq |\beta| \leq s \) then exists \( C_\beta \) such that \(|x^\beta| \leq C_\beta (1 + |x|^2)^{s/2} \).

**Exercise 8** Show that if \( f \in \mathcal{S} \) then \( f \in L^1(\mathbb{R}^n) \).