The open mapping theorem

**Theorem** Open mapping theorem Let $X$ and $Y$ be Banach spaces and suppose that the operator $L \in \mathcal{L}(X, Y)$ is surjective. Then $L(O)$ is open for any open set $O$ in $X$.

**Exercise 1** Show that if $L : X \rightarrow Y$ is bijective and linear then the inverse is also linear.

Recall that the function $f$ is continuous if and only if the inverse image $f^{-1}(O)$ is an open set whenever $O$ is open. With this in mind the following theorem is an important consequence:

**Theorem** Inverse mapping theorem. Let $X$ and $Y$ be Banach spaces and suppose that the operator $L \in \mathcal{L}(X, Y)$ is invertible. Then the inverse is linear and bounded.

**Exercise 2** Prove the inverse mapping theorem.

**Exercise 3** Give example of bounded, linear operator $L \in \mathcal{L}(X, X)$ that is injective but not surjective. Observe that this is not possible if $X$ is finite dimensional. If a quadratic matrix has a left inverse it has automatically a right inverse also and is hence invertible.

**Exercise 4** Let $X$ be a Banach space with two different norms $\| \cdot \|_1$ and $\| \cdot \|_2$ and suppose that there exists a constant $C$ such that $\|x\|_1 \leq C \|x\|_2$. Show that the norms are equivalent.

**Exercise 5** Show that the identity operator from $C([0, 1], \| \cdot \|_{\infty})$ to $C([0, 1], \| \cdot \|_1)$ is a bounded operator but that the identity operator from $C([0, 1], \| \cdot \|_1)$ to the space $C([0, 1], \| \cdot \|_{\infty})$ is unbounded. Explain why this does not contradict the inverse mapping theorem.

**Principle of uniform boundedness**

**Theorem** Principle of uniform boundedness. Let $X$ and $Y$ be Banach spaces and let $A$ be a family of bounded linear operators from $X$ to $Y$. If, for every $x \in X$,

$$\sup_{L \in A} \|L(x)\|_Y < \infty$$

then

$$\sup_{L \in A} \|L\| < \infty.$$

Also this theorem have alternative formulations. Important is the following:

**Theorem** Banach-Steinhaus. Let $X$ and $Y$ be Banach spaces and let $L_n \in \mathcal{L}(X, Y)$ be
a sequence such that for all \( x \in X \), \( \lim_{n \to \infty} L_n(x) \) exists. Define \( L : X \to Y \) by \( L(x) = \lim_{n \to \infty} L_n(x) \), then \( L \) is a bounded, linear operator.

Exercise 6 Prove the Banach-Steinhaus theorem using the uniform boundedness theorem.

Observe that the theorem gives a pointwise limit of \( L_n \), but it does not say that \( \|L_n - L\| \to 0 \) or \( \|L_n\| \to \|L\| \).

Exercise 7 Consider \( L_n : \ell^2 \to \mathbb{R} \) given by \( L_n(x) = x_n \). Find \( L \) and compute \( \|L_n - L\| \) and \( \|L\| \).

Let \( X \) be a Banach space and let \( x_n \in X \) be a sequence. Recall that the sequence \( x_n \) is weakly convergent to \( x \in X \) if \( \xi(x_n) \to \xi(x) \) for all \( \xi \in X^* \). Now introduce the notation that the sequence \( x_n \) is weakly convergent if the sequence \( \xi(x_n) \) is convergent (to a real number) for all \( \xi \in X^* \). Note that this definition does not include any limit in \( X \). A consequence of Banach-Steinhaus is the following:

**Theorem** Let \( X \) be a reflexive Banach spaces. Then any weakly convergent sequence \( x_n \) in \( X \) has a limit, i.e there exists an \( x \in X \) such that \( x_n \) converges weakly to \( x \).

Exercise 8 Given \( x \in X \) then for \( \xi \in X^* \), \( \hat{x}(\xi) = \xi(x) \) defines a function from \( X^* \) to \( \mathbb{R} \). Show that \( \hat{x} \) is linear, bounded, i.e. \( \hat{x} \in X^{**} \). Also show that \( \|\hat{x}\|_{X^{**}} = \|x\|_X \).

Exercise 9 Show that the mapping \( x \mapsto \hat{x} \) from \( X \) to \( X^{**} \) is linear, injective and isometric.

If the mapping also is surjective then \( X \) is called reflexive and then any element in \( X^{**} \) can be identified with an element in \( X \). Every Hilbert space is reflexive according to the Riesz representation theorem.

Exercise 10 Prove the theorem about that there exists a limit point of a weakly convergent sequence.

On the other hand one says that a sequence \( \xi_n \in X^* \) is weakly *-convergent if \( \xi_n(x) \) converges (to a real number) for all \( x \in X \) and the sequence is weakly *-convergent to \( \xi \in X^* \) if \( \xi_n(x) \) converges to \( \xi(x) \) for all \( x \in X \).

**Theorem** Let \( X \) be a Banach spaces. Then any weakly *-convergent sequence \( \xi_n \) in \( X^* \) has a limit, i.e there exists an \( \xi \in X^* \) such that \( \xi_n \) is weakly *-convergent to \( \xi \).

Note that this theorem does not require that the Banach space is reflexive.

Exercise 11 Prove the theorem.