The open mapping theorem

**Theorem** *Open mapping theorem*. Let $X$ and $Y$ be Banach spaces and suppose that the operator $L \in \mathcal{L}(X, Y)$ is surjective. Then $L(O)$ is open for any open set $O$ in $X$.

**Exercise 1** Show that if $L : X \to Y$ is bijective and linear then the inverse is also linear.

Recall that the function $f$ is continuous if and only if the inverse image $f^{-1}(O)$ is an open set whenever $O$ is open. With this in mind the following theorem is an important consequence:

**Theorem** *Inverse mapping theorem*. Let $X$ and $Y$ be Banach spaces and suppose that the operator $L \in \mathcal{L}(X, Y)$ is invertible. Then the inverse is linear and bounded.

**Exercise 2** Prove the inverse mapping theorem.

**Exercise 3** Give example of bounded, linear operator $L \in \mathcal{L}(X, X)$ that is injective but not surjective. Observe that this is not possible if $X$ is finite dimensional. If a quadratic matrix has a left inverse it has automatically a right inverse also and is hence invertible.

**Exercise 4** Let $X$ be a Banach space with two different norms $\| \cdot \|_1$ and $\| \cdot \|_2$ and suppose that there exists a constant $C$ such that $\| x \|_1 \leq C \| x \|_2$. Show that the norms are equivalent.

**Exercise 5** Show that the identity operator from $C([0, 1], \| \cdot \|_{\infty})$ to $C([0, 1], \| \cdot \|_1)$ is a bounded operator but that the identity operator from $C([0, 1], \| \cdot \|_1)$ to the space $C([0, 1], \| \cdot \|_{\infty})$ is unbounded. Explain why this does not contradict the inverse mapping theorem.

**Principle of uniform boundedness**

**Theorem** *Principle of uniform boundedness*. Let $X$ and $Y$ be Banach spaces and let $A$ be a family of bounded linear operators from $X$ to $Y$. If, for every $x \in X$,

$$\sup_{L \in A} \| L(x) \|_Y < \infty$$

then

$$\sup_{L \in A} \| L \| < \infty.$$

Also this theorem have alternative formulations. Important is the following:
**Theorem**  **Banach-Steinhaus.** Let $X$ and $Y$ be Banach spaces and let $L_n$ be a sequence of bounded linear operators from $X$ to $Y$. Assume that for all $x \in X$, $\lim_{n \to \infty} L_n(x)$ exists. Define $L : X \to Y$ by $L(x) = \lim_{n \to \infty} L_n(x)$, then $L$ is a bounded, linear operator.

**Exercise 6** Prove the theorem.

Observe that the theorem gives a pointwise limit of $L_n$, but it does not say that $\|L_n - L\| \to 0$ or $\|L_n\| \to \|L\|$.

**Exercise 7** Consider $L_n : \ell^2 \to \mathbb{R}$ given by $L_n(x) = x_n$. Find $L$ and compute $\|L_n - L\|$, $\|L_n\|$, and $\|L\|$.

Let $X$ be a Banach space and let $x_n \in X$ be a sequence. Recall that the sequence $x_n$ is weakly convergent to $x \in X$ if $\xi(x_n) \to \xi(x)$ for all $\xi \in X^*$. Now introduce the notation that the sequence $x_n$ is weakly convergent if the sequence $\xi(x_n)$ is convergent (to a real number) for all $\xi \in X^*$. Note that this definition does not include any limit in $X$. A consequence of Banach-Steinhaus is the following:

**Theorem** Let $X$ be a reflexive Banach spaces. Then any weakly convergent sequence $x_n$ in $X$ has a limit, i.e there exists an $x \in X$ such that $x_n$ converges weakly to $x$.

**Exercise 8** Given $x \in X$ then for $\xi \in X^*$, $\hat{x}(\xi) = \xi(x)$ defines a function from $X^*$ to $\mathbb{R}$. Show that $\hat{x}$ is linear, bounded, i.e. $\hat{x} \in X^{**}$. Also show that $\|\hat{x}\|_{X^{**}} = \|x\|_X$.

**Exercise 9** Show that the mapping $x \mapsto \hat{x}$ from $X$ to $X^{**}$ is linear, injective and isometric.

If the mapping also is surjective then $X$ is called reflexive and then any element in $X^{**}$ can be identified with an element in $X$. Every Hilbert space is reflexive according to the Riesz representation theorem.

**Exercise 10** Prove the theorem about that there exists a limit point of a weakly convergent sequence.

On the other hand one says that a sequence $\xi_n \in X^*$ is weakly $^*$-convergent if $\xi_n(x)$ converges (to a real number) for all $x \in X$ and the sequence is weakly $^*$-convergent to $\xi \in X^*$ if $\xi_n(x)$ converges to $\xi(x)$ for all $x \in X$.

**Theorem** Let $X$ be a Banach spaces. Then any weakly $^*$-convergent sequence $\xi_n$ in $X^*$ has a limit, i.e there exists an $\xi \in X^*$ such that $\xi_n$ is weakly $^*$-convergent to $\xi$.

Note that this theorem does not require that the Banach space is reflexive.

**Exercise 11** Prove the theorem.