Notions of convergence

In functional analysis we will work with several notions of convergence. Some of the more important for real valued functions are the following:

**Pointwise convergence.** The sequence $f_n \to f$ pointwise when $n \to \infty$, if for every $x$, $f_n(x) \to f(x)$.

**Uniform convergence.** The sequence $f_n \to f$ uniformly when $n \to \infty$, if $\sup_x |f_n(x) - f(x)| \to 0$. If the functions are continuous this can be written $\|f_n - f\|_\infty \to 0$.

**Convergence in $L^p$.** $f_n \to f$ in $L^p$ when $n \to \infty$, if $\|f_n - f\|_p \to 0$.

**Exercise 1** Show that if $f_n \to f$ uniformly on $[0,1]$, then $f_n \to f$ pointwise and in $L^2(0,1)$.

**Exercise 2** Define

$$f_n(x) = \frac{n}{1 + n^2 x^2}.$$  

What is the pointwise limit of $f_n$? Show that this is not an $L^2(0,1)$ limit. Does $f_n$ converge uniformly?

**Hint:**

$$\int \frac{1}{(1 + y^2)^2} \; dy = \frac{1}{2} \left( \frac{y}{1 + y^2} + \arctan y \right).$$

**Exercise 3** Define

$$g_n(x) = \begin{cases} n, & \text{if } |x| < 1/n^2, \\ 0, & \text{otherwise}. \end{cases}$$

Show that $g_n \to 0$ in $L^2(-1,1)$. Is the convergence pointwise? Does $g_n$ converge to 0 uniformly?

**Exercise 4** Let

$$\phi(x) = \begin{cases} 1, & 0 < x < 1/2, \\ 0, & 1/2 < x < 1, \end{cases}$$

be a 1-periodic function, and define $\phi_n(x) = \phi(nx)$. Show that

$$\int_a^b \phi_n(x) - 1/2 \, dx \to 0, \quad \text{when } n \to \infty,$$

for any interval $(a, b)$. 


The Riemann-Lebesgue lemma

Theorem (The Riemann-Lebesgue lemma.) If $f \in L^1([0, 2\pi])$ with Fourier coefficients $c_k(f)$, then $\lim_{|k| \to \infty} c_k = 0$, i.e.

$$\frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, dx \to 0$$

as $|k| \to \infty$.

Exercise 5 Prove the Riemann-Lesbesgue lemma if $f \in L^2([0, 2\pi])$.

Exercise 6 Is it true that $L^2([0, 1]) \subseteq L^1([0, 1])$? Give a proof or a counterexample. Is $L^1([0, 1]) \subseteq L^2([0, 1])$? Is $L^2([1, \infty)) \subseteq L^1([1, \infty))$?

Let

$$P(x) = \sum_{k=-N}^{N} c_k e^{ikx}$$

denote a trigonometric polynomial of degree $N$.

Exercise 7 Convince yourself that the trigonometric polynomials are dense in $L^1([0, 2\pi])$ (Hint: Study the lecture notes and use the definition of $L^1$).

Exercise 8 Compute the Fourier coefficients $c_k(f)$ of $f(x) = e^{inx}, 0 \leq x \leq 2\pi$.

Exercise 9 If $P(x)$ is a trigonometric polynomial of degree $N$, what can you say about the Fourier coefficients $c_k$ when $|k| > N$?

Exercise 10 Prove the Riemann-Lebesgue lemma.