Projections

Let $X$ be a linear space and let $M, N$ be linear subspaces. Let the sum of $M$ and $N$ be the set

$$M + N = \{m + n ; m \in M, n \in N\}$$

and write the sum as a direct sum $M \oplus N$ if the decomposition of any element as $m + n$ with $m \in M$ and $n \in N$ is unique.

Exercise 1
Show that $M + N$ is a linear subspace of $X$.

A linear mapping $P : X \to M$ is called a projection if $P^2 = P$.

Exercise 2
Show that if $X = M \oplus N$ then the mapping $P : X \to M$ for $x = m + n$ given by $P(x) = m$ is a projection (along $N$). And on the other hand, if $P$ is a projection then, with $M = P(X)$ and $N = \text{Ker} P$, $X = M \oplus N$.

Orthonormal sets

Exercise 3
Show that $\varphi_k(x) = H(x - k) - H(x - k - 1)$ for $k = 0, 1, 2, \ldots$, where $H$ is the Heaviside function, is an orthonormal set in $L^2(\mathbb{R}_+).$ Find the projection $p$ of $f(x) = e^{-x \ln 2} \cdot \ln 2$ on the linear hull of the orthonormal set. Compute $\|p - f\|_2$. Is the orthonormal set an orthonormal basis?

Linear operators

Exercise 4
Define $S$ on $\ell^2$ by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

Show that $S$ is a bounded linear operator. Is $S$ injective? Surjective?

Exercise 5
Define $S^*$ on $\ell^2$ by

$$S^*(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Show that $S^*$ is a bounded linear operator. Is $S^*$ injective? Surjective? Compute $SS^*$ and $S^*S$. 
The space of bounded linear mappings $\mathcal{L}(X, Y)$

Let $X$ and $Y$ be Banach spaces. Recall that a linear mapping $L : X \to Y$ is said to be bounded if there exists a constant $C$ such that

$$\|L(x)\|_Y \leq C\|x\|_X$$

for all $x \in X$. (1)

We will show that the space of bounded linear mappings from $X$ to $Y$, denoted by $\mathcal{L}(X, Y)$, is in fact also a Banach space, with the norm

$$\|L\| = \sup_{\|x\| \neq 0} \frac{\|L(x)\|_Y}{\|x\|_X}.$$  

(2)

The definition of the norm gives the useful estimate

$$\|L(x)\|_Y \leq \|L\|\|x\|_X$$

for all $x \in X$. (3)

On the other hand, if the linear map $L$ satisfies (1) then $\|L\| \leq C$.

**Exercise 6** Show that $\mathcal{L}(X, Y)$ is a linear space, if we define

$$(L + M)(p) = L(p) + M(p)$$

$$(aL)(p) = aL(p)$$

for $a \in \mathbb{R}$, $L, M \in \mathcal{L}(X, Y)$, $p \in X$.

**Exercise 7** Show that $\| \cdot \|$ given by (2) is a norm on $\mathcal{L}(X, Y)$.

**Exercise 8** Show that any Cauchy sequence $\{z_n\}$ in a normed linear space $Z$ is bounded, i.e., there exists a constant $C$ such that $\|z_n\| \leq C$ for all $n$.

**Exercise 9** Show that $\mathcal{L}(X, Y)$ is complete, i.e that every Cauchy sequence in $\mathcal{L}(X, Y)$ converges in the norm given by (2).