Hilbert space geometry

Let $H$ be a Hilbert space and let $A$ be a subset of $H$. We define the orthogonal complement by

$$A^\perp = \{ h \in H; (h, y) = 0 \text{ for all } y \in A \}.$$  

**Exercise 1** Show that $A^\perp$ is a linear subspace of $H$.

While discussing if subset are closed it is useful to recall the following notion. A set $A$ is **sequentially closed** if $x_n \to x, \ x_n \in A$ implies $x \in A$. In a metric space this is equivalent to that the set $A$ is closed.

**Exercise 2** Show that $A^\perp$ is closed.

That all linear subspaces are closed is typical for the finite dimensional spaces.

**Exercise 3** Show that a finite dimensional subspace is closed.

**Exercise 4** Give example of a linear subspace that is not closed.

Another way of proving that a set is closed is to use that the inverse image of a closed set is closed when dealing with continuous functions.

**Exercise 5** If $B$ is a Banach space show that the unit sphere $\{ x \in B; \|x\| = 1 \}$ is closed. Is the unit ball closed? (Recall that $f(x) = \|x\|: B \to \mathbb{R}$ is continuous.)

**Theorem** **Nearest point property.** Let $M$ be a nonempty, closed, convex subset of a Hilbert space $H$. Then given any $h \in H$ there exist a unique $m_0 \in M$ such that

$$\|h - m_0\| = \inf_{m \in M} \|h - m\|.$$  

It is important that the set $M$ is both closed and convex.

**Exercise 6** Give an example that the theorem is not valid if $H$ is a Banach space.

**Theorem** **Projection theorem.** Let $M$ be a closed subspace of a Hilbert space $H$. Then ever $h \in H$ can be written uniquely as $h = m + m_\perp$ with $m \in M$ and $m_\perp \in M^\perp$.

**Exercise 7** If $M$ is a linear subspace of $H$, what is the relation between $M$ and $M^{\perp \perp}$?
**Exercise 8**  If $M$ is a proper, closed linear subspace of a Hilbert space $H$, show that there exists a vector $y \neq 0$ that is normal to $M$.

**Exercise 9**  If there is no nonzero vector $y$ that is normal to a closed linear subspace $M$ of a Hilbert space $H$, show that $M = H$.

**Linear functionals**

Let $U$, $V$ be linear spaces. A function $L: U \to V$ is called a linear function (transformation, mapping, operator) if

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L(u_1) + \alpha_2 L(u_2) \quad \text{for all } \alpha_1, \alpha_2 \in \mathbb{R}, \text{ all } u_1, u_2 \in U.$$ 

Let $X$, $Y$ be normed, linear spaces. A linear function $L: X \to Y$ is bounded if there exists $C > 0$ such that

$$\|L(x)\|_Y \leq C\|x\|_X, \quad \text{for all } x \in X.$$ 

If $Y = \mathbb{R}$ the linear function is also called a linear functional. (Sometimes boundedness is also included in the notation.)

**Theorem**  **Riesz representation theorem.**  If $L$ is a bounded linear functional on a Hilbert space $H$, then there exists a unique $u \in H$ such that $L(x) = (x, u)$.

**Exercise 10**  If $(h, u) = (h, v)$ for all $h \in H$, show that $u = v$.

**Exercise 11**  Which of the following are functionals? Which are linear?

a) $f: x \mapsto \|x\|_X$ for $x \in X$, where $X$ is a normed linear space.

b) $f: x \mapsto \int_0^1 x(t) \, dt$ for $x \in C[0, 1]$.

c) Let $x = (x_1, x_2, \ldots) \in \ell^2$ and define $f: x \mapsto (0, x_1, x_2, \ldots)$.

d) $f: x \mapsto x'(1/2)$ for $x \in C^1[0, 1]$.

e) Let $c \in \ell^{\infty}$ and define $f: x \mapsto \sum_{n=1}^{\infty} c_n x_n$ for $x \in \ell^1$.

**Exercise 12**  Prove the following theorem (Hahn-Banach for Hilbert spaces). Let $H$ be a Hilbert space and let $M$ be a linear subspace of $H$. Let $\xi_M$ be a bounded linear mapping from $M$ to $\mathbb{R}$. Then there is a linear functional $\xi \in H^*$ such that $\xi|_M = \xi_M$ and $\|\xi\| = \|\xi_M\|$. 