Hilbert space adjoints

If \( H \) is a Hilbert space and if \( A \in \mathcal{L}(H) = \mathcal{L}(H, H) \) then the Hilbert space adjoint \( A^* \) is defined by

\[
(A(x), y) = (x, A^*(y)), \quad \text{for all } x, y \in H.
\]

The Hilbert space adjoint always exists by the Riesz representation theorem.

**Exercise 1**
Check that \( A^* \) is linear and bounded, that is, that \( A^* \in \mathcal{L}(H) \). Also show that \( \|A^*\| = \|A\| \).

**Exercise 2**
If \( A \in \mathcal{L}(H) \), what is \( A^{**} \)?

**Exercise 3**
If \( A \in \mathcal{L}(H) \), show that \( \ker A = (\operatorname{Im} A^*)^\perp \). Also show that \( A \) is injective precisely if \( A^* \) is surjective.

Spectral theory

**Exercise 4**
Let the left shift operator \( L : \ell^2 \to \ell^2 \) be given by

\[
L(x) = (x_2, x_3, x_4, \ldots).
\]

Show that the spectrum is contained in the closed unit disk and compute the point spectrum.

**Exercise 5**
Let the multiplication operator \( M : C_b([0, 1]) \to C_b([0, 1]) \) be defined by

\[
M(u)(x) = xu(x).
\]

Describe \( \text{sp}(M) \).

**Exercise 6**
Let

\[
\mathcal{D}(A) = \{ u \in H^2(0, 1) ; u(0) = u(1) = 0 \}
\]

and define the operator \( (\mathcal{D}(A), A) \) from \( L^2(0, 1) \) to \( L^2(0, 1) \) by

\[
Au = u'', \quad \text{for } u \in \mathcal{D}(A).
\]

Show that \( \text{sp}(A) \) is not compact.

**Exercise 7**
Let \( A \in \mathcal{L}(X, X) \) where \( X \) is an Banach space. Show that

\[
\|R(A)\| \to 0, \quad \text{as } \lambda \to \infty.
\]