Operators

Let $A_n$ and $A$ be bounded operators $X \to Y$ on some Banach spaces. We say that $A_n \to A$ in operator norm if $\|A_n - A\| \to 0$ and we say that $A_n \to A$ strongly if $A_n(x) \to A(x)$, i.e., $\|A_n(x) - A(x)\|_Y \to 0$ for any $x \in X$.

Exercise 1
Let $A_n : \ell^2 \to \ell^2$ be given by

$$A_n(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots).$$

Show that $A_n \to I$, $I$ the identity operator, strongly but not in operator norm.

Exercise 2
Let $R : \ell^2 \to \ell^2$ and $L : \ell^2 \to \ell^2$ be given by

$$R(x) = (0, x_1, x_2, \ldots)$$

and

$$L(x) = (x_2, x_3, x_4, \ldots).$$

What is $RL$ and $LR$? Compute the range, nullspace and norm of $L$ and $R$. Is any of them invertible?

Exercise 3
If $X$, $Y$ are Banach spaces and $(D(A), A)$ is a linear operator from $X$ to $Y$ with range $\mathcal{R}(A)$. Show that the inverse $(\mathcal{R}(A), A^{-1})$ exists if and only if $\mathcal{N}(A) = \{0\}$. Also show that the inverse is linear if it exists.

Exercise 4
Let $X$, $Y$ and $Z$ be Banach spaces and let $A : X \to Y$ and $B : Y \to Z$ be bijective operators. Let $BA : X \to Z$ be the composition. Show that

$$(BA)^{-1} = A^{-1}B^{-1}.$$
Find the nullspace and the range, show that \( I \) is bounded and compute the norm.

**Exercise 7** A mapping is open if it maps every open set to an open set. Show that an open mapping need not map closed sets to closed sets.

**Exercise 8** Let \( X = \ell_0 \) be the set of sequences \( x = (x_1, x_2, x_3, \ldots) \) with only finitely many nonzero coordinates and with norm
\[
\|x\| = \sup_k |x_k|.
\]

Let \( T : X \to X \) be defined by
\[
T(x) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots).
\]

Show that \( T \) is linear and bounded but that \( T^{-1} \) is unbounded. Why does this not contradict the bounded inverse theorem?

Assume that \( (X, \| \cdot \|_X), (Y, \| \cdot \|_Y) \) are Banach spaces. Consider the product space \( X \times Y \) with norm
\[
\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y.
\]

**Exercise 9** Show that this indeed is a norm, and that \( X \times Y \) with this norm is a Banach space.

If \( T : X \to Y \) is linear, the subset
\[
\text{graph}(T) = \{(x, T(x)) \in X \times Y ; x \in X\}
\]
of \( X \times Y \) is called the **graph** of \( T \).

**Exercise 10** Show that the property that \( \text{graph}(T) \) is closed (in \( X \times Y \)) is equivalent to the statement that: whenever \( x_n \to x \) and \( T(x_n) \to y \) then \( y = T(x) \).

**Exercise 11** Show that if \( T \in \mathcal{L}(X, Y) \), then \( \text{graph}(T) \) is closed.

This last exercise is the only if part of the following theorem, where the if part is equivalent to the bounded inverse theorem.

**Theorem** **Closed graph theorem.** Let \( X \) and \( Y \) be Banach spaces and let \( T : X \to Y \) be linear, then \( T \) is bounded if and only if \( \text{graph}(T) \) is closed in \( X \times Y \).