Sobolev spaces on $\mathbb{R}^n$

The property that $C^\infty_0(\Omega)$ is dense in $H'(\mathbb{R}^n)$ for all $s > 0$ is often useful while showing statements about functions in $H'(\mathbb{R}^n)$.

**Exercise 1** For $u \in H'(\mathbb{R}^n)$ with $s > n/2$, show that $\lim_{x \to \infty} u(x) = 0$.

**Exercise 2** Show that the Sobolev imbedding theorem fails in general if $\partial \Omega$ has a cusp by considering a function that has a singularity at the cusp. For example, consider $\Omega = \{(x_1, x_2) \in \mathbb{R}^2; 0 < x_2 < x_1^a, 0 < x_1 < 1\}$ and functions like $u(x_1, x_2) = x_1^c$ for suitable $a$ and $c$ such that $u \in H^2(\Omega)$.

The following exercise outlines a general method to show that for an open, bounded set $\Omega \subset \mathbb{R}^n$ and $k \geq 1$, $C^\infty_0(\Omega)$ is not dense in $H^k(\Omega)$ or $W^{k,p}(\Omega)$ and, hence, that $H^k_0(\Omega)$ is a proper subset of $H^k(\Omega)$.

**Exercise 3** Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}$ and consider for a fixed $u \in H^1(\Omega)$ the operator $L_u : H^1(\Omega) \to \mathbb{R}$ given by

$$L_u(\varphi) = (\varphi, u)_{H^1} = \int_{\Omega} \varphi u \, dx + \int_{\Omega} \varphi' u' \, dx.$$ 

Show that $L_u$ is a linear, bounded functional.

Show that if $\varphi \in C^\infty_0(\Omega)$, the functional can also be written

$$L_u(\varphi) = \int_{\Omega} \varphi (u - u'') \, dx$$

and show that there exists $u \in H^1(\Omega)$, $u \neq 0$, such that $u - u'' = 0$. Hence, $L_u(\varphi) = 0$ for all $\varphi \in C^\infty_0(\Omega)$ but $L_u(u) > 0$. Show that this implies that $C^\infty_0(\Omega)$ is not dense in $H^1(\Omega)$. Why can $C^\infty_0(\mathbb{R})$ still be dense in $H^1(\mathbb{R})$?