Norms and metrics

If $M$ is a non-empty set a *metric*, also called a distance function, is a real-valued function $d(x, y)$ satisfying for $x, y, z \in M$

- positivity: $d(x, y) \geq 0$ and $d(x, y) = 0$ precisely if $x = y$,
- symmetry: $d(x, y) = d(y, x)$ and
- the triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

A non-empty set $X$ is called a (*real*) linear space or a vector space if elements $x, y \in X$ may be added, denoted $x + y$ and be multiplied by real numbers $\alpha \in \mathbb{R}$ called scalars, multiplication denoted $\alpha x$ and in both cases giving new elements of $X$. The addition an multiplikations satisfies several well-known rules. Observe that all linear spaces contains a special element 0 called origo. If the scalars instead ar complex numbers the linear space is called a complex linear space. In the following it is assumed that the scalars are real unless otherwise is stated explicitly.

A norm on a linear space $X$ is a real-valued function $\| \cdot \| : X \to \mathbb{R}$ satisfying

- positivity: $\| x \| \geq 0$ and $\| x \| = 0$ precisely if $x = 0$,
- scaling property: $\| \alpha x \| = |\alpha| \| x \|$ and
- the triangle inequality: $\| x + y \| \leq \| x \| + \| y \|$.

A linear space together with a norm is called a normed linear space.

**Exercise 1** If $X$ with norm $\| \cdot \|$ is a normed linear space show that $d(x, y) = \| x - y \|$ is a metric on $X$.

**Exercise 2** In $\mathbb{R}^2$ find a metric that is not given by a norm, i.e., find a metric $d(x, y)$ and show that there exists no norms such that $d(x, y) = \| x - y \|$ for $x, y \in \mathbb{R}^2$.

For $x \in \mathbb{R}^n$, $n \in \mathbb{Z}_+$, define the norms

$$\| x \|_p = \left( \sum_{k=1}^n |x_k|^p \right)^{1/p}, \quad \text{and} \quad \| x \|_\infty = \max_{1 \leq k \leq n} |x_k|.$$ 

These can be generalized to infinite sequences $x = \{x_k\}_{k=1}^\infty$ as

$$\| x \|_p = \left( \sum_{k=1}^\infty |x_k|^p \right)^{1/p}, \quad \text{and} \quad \| x \|_\infty = \max_{k \in \mathbb{Z}_+} |x_k|.$$
Now let $\ell^p$ denote the set of sequences $x$ such that $\|x\|_p < \infty$ and correspondingly for $\ell^\infty$. These are (can be shown to be) simple examples of infinite dimensional Banach spaces.

**Exercise 3** Check that $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^2$. To see that they are different compute the distance $d((2,1), (5,5))$ in each of the three norms.

Observe that if $a$ and $b$ are real numbers satisfying $a \leq b$ and $-a \leq b$ then $|a| \leq b$.

**Exercise 4** For any normed linear space $X$ with norm $\|\cdot\|$, prove the reverse triangle inequality

$$|||x|| - ||y||| \leq \|x - y\|, \quad x, y \in X.$$

**Inner product spaces**

Let $X$ be an inner product space, i.e., a linear space $X$ over $\mathbb{R}$ (or $\mathbb{C}$), with inner product is a real-valued function $(\cdot, \cdot) : X \times X \to \mathbb{R}$ (or $\mathbb{C}$) satisfying

- linearity: $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$,
- symmetry: $(x, y) = \overline{(y, x)}$,
- positivity: $(x, x) \geq 0$ and $(x, x) = 0$ precisely if $x = 0$.

For $x, y \in X$ the vector $\frac{(x, y)}{(y, y)} y$ is called the projection of $x$ on $y$.

**Exercise 5** Show that $y$ and $x - \frac{(x, y)}{(y, y)} y$ are orthogonal.

**Theorem** Cauchy-Bunyakovsky-Schwarz inequality.

$$|(x, y)|^2 \leq (x, x)(y, y), \quad x, y \in X.$$

**Exercise 6** Show this by expanding $\left(x - \frac{(x, y)}{(y, y)} y, x - \frac{(x, y)}{(y, y)} y\right)$.

If $X$ is an inner product space the inner product gives a compatible norm by

$$\|x\| = (x, x)^{1/2}.$$

**Exercise 7** Check that this is a norm.
**Corollary** After these checks the inequality may be rewritten as **Cauchy-Bunyakovsky-Schwarz inequality.**

\[ |(x, y)| \leq \|x\|\|y\|, \quad x, y \in X. \]

**Theorem** Every norm that originates from an inner product satisfies the **parallelogram law**:

\[ \|x + y\|^2 + \|x - y\|^2 = 2 (\|x\|^2 + \|y\|^2) . \]

**Exercise 8** Prove this.

**Exercise 9** Check if the parallelogram law is valid in \( \ell^1 \) and \( \ell^\infty \).

**Continuity**

Recall that if \( X, Y \) are metric spaces with metrics \( d_X \) and \( d_Y \) respectively, then a function \( f : X \to Y \) is **continuous at** \( y \in X \) if there for any \( \varepsilon > 0 \) exists \( \delta > 0 \) such that

\[ d_Y(f(x), f(y)) < \varepsilon \quad \text{whenever} \ d_X(x, y) < \delta \]

and \( f \) is **continuous** if it is continuous at all \( y \in X \). Furthermore, \( f \) is **Lipschitz continuous with Lipschitz constant** \( L \) if

\[ d_Y(f(x), f(y)) \leq L \, d_X(x, y), \quad \text{for all} \ x, y \in X. \]

**Exercise 10** Show that a Lipschitz continuous function is continuous and that a continuous function is not necessarily Lipschitz continuous.

Next we study the continuity properties of norms and inner products.

**Exercise 11** If \( X \) is a normed linear space, show that the mapping \( x \mapsto \|x\| \) is continuous.

**Exercise 12** If \( X \) is an inner product space, show that the mapping \( x \mapsto (x, y) \) is linear (or anti linear) and continuous for all \( y \in X \).

**Exercise 13** Let \( X, Y \) be normed linear spaces and let \( f : X \to Y \) be a continuous function that satisfies

\[ f(x + y) = f(x) + f(y), \quad \text{for all} \ x, y \in X, \ (f \text{ is said to be additive}). \]

Show that

\[ f(\alpha x) = \alpha f(x), \quad \text{for all} \ x \in X \text{ and} \ \alpha \in \mathbb{R}, \ (i.e. \ f \text{ is linear}). \]
Let $X$ be a normed linear space over $\mathbb{R}$ with norm $\|x\|$ satisfying the parallelogram law. Show that $(x, y)$ defined by

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2),$$

for all $x, y \in X$, is an inner product.