5.48 Volyms pyramid = \( \frac{1}{3} \cdot \text{basis} \cdot \text{höjd} \)

\[ V = \frac{1}{3} \cdot 2 \cdot 2y \cdot (2-2) = \frac{4}{3} y, (2-2), \text{ så} \]

maximal \( f(x, y, z) = y (2-z) \) under

Bivillkor \( g(x, y, z) = x^2 + \frac{y^2}{4} + (z-1)^2 = 1 \).

Bivillkret "kompakt och saknar rand", så vi

daterar bara inre punktor: \( \text{grad } f \parallel \text{grad } g \).

\[ \text{grad } f = (g(2-z), x(2-z), -xy), \text{ grad } g = (2x, \frac{y}{2}, 2(z-1)) \]

\[ \begin{align*}
  y(2-z) &= 1 & \Rightarrow & (2-z) &= \frac{12}{5} & \Rightarrow & x &= \frac{12}{5} \\
  2(2-z) &= 1 \frac{y}{2} & \Rightarrow & x + \frac{y^2}{4} + (z-1)^2 &= 1 & \Rightarrow & (z-1)^2 &= 1 \\
  -xy &= 1 & \Rightarrow & z &= 2(z-1) & \Rightarrow \\
\end{align*} \]

\( y(2-z) \neq 0 \)

\( x^2 = y^2 \Rightarrow y = 2x \) (uppenbart \( x > 0, y > 0 \Rightarrow 170 = 2 \cdot 2 \))

\( \Rightarrow 1 = 2z^2 - 2 \Rightarrow 2z^2 - 2 = 1 \Rightarrow 2z^2 = 3 \Rightarrow z^2 = \frac{3}{2} \Rightarrow z = \frac{\sqrt{3}}{2} \)

\( x^2 = y^2 \Rightarrow 2x^2 = 2(2-z)(2-1) \Rightarrow x^2 = (2-z)(2-1) \Rightarrow \)


\[ x + \frac{(2x)^2}{4} + (x-1)^2 = \frac{x^2}{2} + (x-1)^2 = 1 \quad (\Rightarrow) \]

\[ 2(z-2)(z-1) + (z-1) = 2(z^2 - 3z + 2) + z^2 - 2z + 1 = \]

\[ 3z^2 - 8z + 5 = 1 \quad (\Rightarrow) \quad 3z^2 - 8z + 4 = 0 \quad (\Rightarrow) \]

\[ z^2 - \frac{8}{3} z + \frac{4}{3} = 0 \quad (\Rightarrow) \quad z = \frac{4 \pm \sqrt{16 - 4 \cdot \frac{4}{3} \cdot \frac{4}{3}}}{2} = \frac{4 \pm \sqrt{16 - \frac{16}{3}}}{2} = \frac{4 \pm \frac{4}{\sqrt{3}}}{2} = \frac{2 \pm \sqrt{3}}{3} \]

\[ \Rightarrow z = 2, \quad z = \frac{\sqrt{3}}{3} \]

\[ \text{Ett} \quad 2 \text{ ger } v = 0 \quad \text{sättar.} \]

\[ z = \frac{\sqrt{3}}{3} \quad \Rightarrow \quad x = \left( \frac{\sqrt{3}}{3} - 2 \right) \left( \frac{2}{3} - 1 \right) = \frac{\sqrt{3}}{3} \quad (\Rightarrow) \quad x = \frac{2}{3} \]

\[ \Rightarrow \quad y = 2 \cdot \frac{2}{3} = \frac{4}{3} \quad \text{och} \]

\[ f \left( \frac{2}{3}, \frac{4}{3}, \frac{2}{3} \right) = \frac{\sqrt{2}}{2} \cdot \frac{2}{3} \cdot \left( 2 - \frac{2}{3} \right) = \frac{\sqrt{2}}{3} \cdot \frac{4}{3} = \frac{\sqrt{2}}{3} \cdot \frac{4}{3} = \frac{32}{27} \]

Undantagspunkt: grad \( g = (2x, \frac{2}{3}, 2(z-1)) = (0, 0, 0) \)

\[ \Rightarrow \quad (x, y, z) = (0, 0, 1) \quad \text{uppfyller ej avstånd.} \]

Således, största värde i \( f = \frac{32}{27} \cdot \frac{128}{81} = \frac{128}{81} \)

Svar, \( \frac{128}{81} \)
5.56. Paralleltrapets - förhåning 7 7
med båda sidor parallella.

Ett figur: \( A \cos = x^2 + 2 \frac{y^2}{2} = (x+y)^2 \), och vi
skall ha \( x + 2z = x + 2\sqrt{y^2 + z^2} = 6 \) (enhet dm)

\[ \Rightarrow \text{optimering } f(x,y,z) = (x+y)^2 \text{ under } \]

bivillkor \( g(x,y,z) = x + 2\sqrt{y^2 + z^2} = 6 \)

"Naturlig" definierat om: \[ 0 \leq x \leq 6 \]

och alla "gränser" utan \( 0 \leq y \leq 3 \)

\( x = 0 \) 
\( y = 0 \) ger \( f(x,y,z) = 0 \) (Tank !)

\[ \text{grad } f = (2x, 2y, 2z) \], \( \text{grad } g = \left( \frac{2y}{\sqrt{y^2 + z^2}}, \frac{2z}{\sqrt{y^2 + z^2}}, \frac{-2x}{\sqrt{y^2 + z^2}} \right) \]

\[ \text{grad } f / \text{grad } g : \begin{cases} z = d \lambda & (1) \\
\quad \frac{z}{\sqrt{y^2 + z^2}} = \lambda & (2) \\
\quad x + 2\sqrt{y^2 + z^2} = 6 & (3) 
\end{cases} \]
1. ger: \[ d = \frac{2y}{\sqrt{y^2 + z^2}} \Rightarrow \sqrt{y^2 + z^2} = 2y \]

\[ y^2 + z^2 = \frac{4y^2}{4y^2} \Rightarrow y^2 + z^2 = \frac{4y^2}{4} \Rightarrow z = \sqrt{3}y \]

2. \[ x = 6 - 2\sqrt{3}y^2 \Rightarrow 6 - 2y = 6 - 4y \]

Ger nu: \[ 6 - 4y + y = \frac{2y}{y} \Rightarrow 6 - 3y = \frac{2y^2}{3} \Rightarrow 3y = 6y \Rightarrow y = 1 \]

\[ z = \sqrt{3} \cdot 1 = \sqrt{3} \]

och \[ x = 6 - 4 \cdot 1 = 2 \]

Ger \( (x,y,z) = (2,\sqrt{3},1) \) och \( f(2,\sqrt{3},1) = (2 + 1)\sqrt{3} = 3\sqrt{3} \)

Grad g. för så vi har några underlagsspunkter.

Gränsen till \( D_f \) som inte ger \( f(2,\sqrt{3},1) = 0 \): \[ y = 0 \Rightarrow x + 2\sqrt{z^2} = 6 \Rightarrow z = \frac{6-x}{2} \]

\[ f(x,0,2) = \frac{1}{2}(6x - x^2) = h(x) \Rightarrow h(x) = \frac{1}{2}(6 - 2x) = 0 \Rightarrow x = 3 \] och \( h(3) = \frac{1}{2}(18 - 9) = \frac{9}{2} \). Gränsen \( x = 0 \) ger \( h(0) = 0 \). (vi vet sedan hela är) \( x = 6 \) ger avstång 0
\( x = 0 \) 

\[ f(x,y,z) = 2\sqrt{x^2 + z^2} = 6 \quad \Rightarrow \quad 4(\sqrt{x^2 + z^2}) = 36 \quad \Rightarrow \quad x^2 + z^2 = \frac{36}{4} = 9 \quad \Rightarrow \]

\[ y^2 = 9 - x^2 \quad \quad \quad \Rightarrow \quad \quad \quad y = \sqrt{9 - x^2} \quad \Rightarrow \]

\[ f(x,y,\sqrt{9 - x^2}) = 2\sqrt{9 - x^2} = m(x) \quad \Rightarrow \]

\[ m'(x) = \frac{2}{\sqrt{9 - x^2}} \quad \Rightarrow \quad 2 \sqrt{9 - x^2} = 0 \quad \Rightarrow \quad 9 - x^2 = 0 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm \frac{3}{\sqrt{2}} \quad \text{och} \]

\[ m\left(\frac{3}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} \sqrt{9 - \frac{9}{2}} = \frac{3}{\sqrt{2}} \left(\frac{\sqrt{2}}{2}\right) = \frac{3}{2} \]

(Visa att gränsspicka \( z = 0 \) \( z = 3 \) ger \( f(x,y,2) \) = 0)

Tänk på: \( 3\sqrt{3} \approx 3,17 > 3,15 = \frac{9}{2} \quad \Rightarrow \)

Svar: maximal area är \( 3\sqrt{3} \) (dm²)

(Anm: Det är inte så klart om man ska

bivillkorat kompletter, t.ex. genom att skära

det med \( x = 0 \).)
5.64. Lösning av \( z = f(x, y) \) är störst i gradientens riktning; störst \( |\text{grad } f| \).

\( f(x, y) = 10 + x^2 + y^2 \Rightarrow \text{grad } f = \left( 2xy, x \right) \Rightarrow \)

\[ |\text{grad } f| = \sqrt{(2x+y)^2 + x^2} = \sqrt{4x^2 + 4xy + y^2 + x^2} \Rightarrow \]

Maximera \( h(x, y) = 5x^2 + y^2 + 4xy \) då \( x^2 + \frac{y^2}{5} \leq 1 \).

\[
\begin{align*}
\text{Ställ pher.} & \quad \begin{cases} 
  6x + 4y = 0 \\
  4y + 4x = 0 
\end{cases} & \text{makta } x = 0 \\
\text{och } y = 0
\end{align*}
\]

\( h(0, 0) = 0 \)

Randen: Ellipsen: \( y = \sqrt{5} \sin t \), \( 0 \leq t \leq 2\pi \)

\( \Rightarrow x^2 + \frac{y^2}{5} = 1 \), \( \text{föra } f \)

\( h(x, y) = 5 \cos^2 t + 5 \sin^2 t + 4\sqrt{5} \cos t \sin t = \)

\( = 5 + 2\sqrt{5} \sin 2t = h(t) \)

\( h(t) = 4\sqrt{5} \cos 2t = 0 \Leftrightarrow \cos 2t = 0 \Rightarrow \exists t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \)

\( h(\frac{\pi}{2}) = h(\frac{3\pi}{2}) = 5 + 2\sqrt{5} \cdot 1, \quad h(\frac{\pi}{2}) = h(\frac{7\pi}{2}) = 5 + 2\sqrt{5} \cdot 1 \)
Andpunkter: \( h(0) = h(2\pi) = 5 + 2(5.0) = 5 \)

Jämförelse ger att höjden störst = 5+2√5 då:

\[
\begin{align*}
  x &= \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{och} \quad x &= \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \\
  y &= \sqrt{5} \sin \frac{\pi}{4} = \sqrt{5} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{10}}{2} \quad \text{och} \quad y &= \sqrt{5} \sin \frac{3\pi}{4} = -\frac{\sqrt{10}}{2}
\end{align*}
\]

Lutningen = \( \frac{\text{y}}{\text{tangent}} = \sqrt{5} + 2\sqrt{5} \)

Riktningen = \( \frac{\text{y}}{\text{tangent}} = \pm \left( 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{5}}{2} \right) = \pm \frac{\sqrt{10}}{2} (2+\sqrt{5}, 1) \)

Svar: Störst lutning \( \sqrt{5} + 2\sqrt{5} \) i riktning \( \pm \left( \frac{\sqrt{10}}{2} (2+\sqrt{5}, 1) \right) \) från punkten \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{10}}{2} \right) \) och \(-\frac{\sqrt{10}}{2} (2+\sqrt{5}, 1) \) \(-\frac{\sqrt{10}}{2} (2+\sqrt{5}, 1) \)
6.23. Salt $F(x,y) = x^y \sin y$.

Different $x^y \cdot \ln x = e^y \ln y$.

\[ F' = e^y \cdot \ln x + \cos y - x^y \ln x + \cos y, \]

specifically $F'_y(10) = 1.101 \ln 1 + \cos 0 = 1 \neq 0$

\[ \Rightarrow y = g(x) \text{ enlgt Implicit Function Theorem.} \]

Deriving $\cos x^y + \sin g(x) = 1 \Rightarrow$

\[ e^{y \ln x} + \sin g(x) = 1 \text{ map } x: \]

\[ e^{y \ln x} \cdot (g'(x) \ln x + y(x) \cdot \frac{1}{x}) + \cos y \cdot g'(x) \cdot y(x) = 0 \Rightarrow \]

\[ x \cdot g'(x) \ln x + x \cdot y(x) \cdot \frac{1}{x} + \cos y \cdot g'(x) \cdot y(x) = 0 \Rightarrow \]

\[ g'(x) (x \ln x + \cos y(x)) = -x \cdot g'(x) \cdot y(x) \Rightarrow \]

\[ g'(x) = - \frac{x g'(x) y(x)}{x \ln x + \cos y(x)}. \]
7.15. Rita området: \[ y = \frac{1}{2} x \]

"I x-led først:

\[-y \leq x \leq 2y, \quad 0 \leq y \leq 1 \Rightarrow\]

\[
\iint_D \frac{dx \, dy}{1 + (x - 2y)^2} = \int_0^{2y} \left( \int_{-y}^{1} \frac{1}{1 + (x - 2y)^2} \, dx \right) \, dy =
\]

\[
= \int_0^{2y} \left[ \arctan(x - 2y) \right]_{-y}^{1} \, dy = \int_0^{2y} \left( \arctan(1 - 2y) - \arctan(-3y) \right) \, dy =
\]

\[
= \int_0^{2y} \arctan(3y) \, dy = \left[ y \cdot \arctan(3y) \right]_0^{2y} - \int_0^{2y} \frac{3y}{1 + 9y^2} \, dy =
\]

\[
= \arctan(3) - 0 - \left[ \frac{3}{18} \cdot \ln(1 + 9y^2) \right]_0^{2y} = \arctan(3) - \frac{1}{6} \ln 10
\]

Anm: Om y-led först så får vi " dela upp "

i de integralter.